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A TRANSMITTING BOUNDARY FOR THE DYNAMIC FINITE ELEMENT ANALYSIS OF CROSS ANISOTROPIC SOILS

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SUMMARY

The problem of the transmitting boundary used in the dynamic finite element analysis of layered axisymmetric soil models with non-axisymmetric displacements is considered. Each layer is modelled as a homogeneous, viscoelastic cross anisotropic medium with a vertical axis of material symmetry.

INTRODUCTION

Some types of soil and rocks exhibit in their response to stresses a significant degree of material anisotropy. It is a manifestation of their anisotropic fabric and structure acquired during the geological formation process. The mechanical anisotropy is often described as cross anisotropy or transversely isotropy, which indicates the existence of a vertical axis of symmetry and horizontal planes of isotropy.

The dynamic response of a rigid strip foundation on a layered cross anisotropic medium and on a cross anisotropic halfspace have been studied by Gazetas.[1,2] He applied an analytical solution, which is based on a Fourier transformation in the horizontal direction. An analytical solution for the vertical, horizontal and rocking motion of a rigid circular plate on a cross anisotropic elastic halfspace has been given by Kirkner.[3] Both solutions are restricted to materials which fulfill a constraint relationship of the elastic parameters, proposed by Carrier.[4] This relationship reduces the number of elastic parameters from five to four and uncouples the equations of motion. The torsional vibration of a rigid circular plate on a cross anisotropic elastic halfspace has been studied by Constantinou and Gazetas.[5]

Recently, a semi-analytical method for foundations with circular or arbitrary shape on a layered viscoelastic cross anisotropic soil has been presented.[6] The derivation is similar to those of the transmitting boundaries of Waas[7] and Kausel,[8] which are used in the finite element analysis of plane and axisymmetric soil models. In the following, the stiffness matrix of the transmitting boundary for axisymmetric soil models, given originally for isotropic media,[8,9] is extended to cross anisotropic media. The solution is based on the derivation in Reference 8. It does not require a restriction of the elastic parameters. The derivation is given in the frequency domain.

DYNAMIC STIFFNESS MATRIX

A general displacement field in the transmitting element can be expanded in a Fourier series as

$$\mathbf{u}_c = \sum_{n=0}^{\infty} (\boldsymbol{\Phi}_n^s \cdot \mathbf{u}_n^s + \boldsymbol{\Phi}_n^a \cdot \mathbf{u}_n^a) \tag{1}$$

with

$$\mathbf{u}_c = \begin{Bmatrix} u_c \\ w_c \\ v_c \end{Bmatrix} \quad \mathbf{u}_n^s = \begin{Bmatrix} u_n^s \\ w_n^s \\ v_n^s \end{Bmatrix} \quad \mathbf{u}_n^a = \begin{Bmatrix} u_n^a \\ w_n^a \\ v_n^a \end{Bmatrix} \tag{1a}$$

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$$\Phi_n^s = [\cos n\varphi,\ \cos n\varphi,\ -\sin n\varphi] \tag{1b}$$

$$\Phi_n^a = [\sin n\varphi,\ \sin n\varphi,\ \cos n\varphi] \tag{1c}$$

in which u_c, w_c and v_c are the radial, vertical and tangential displacements, respectively. The superscripts 's' and 'a' refer to symmetry and antisymmetry, respectively, about $\varphi = 0$, Figure 1. With this expansion all pertinent equations become uncoupled with respect to n and to 'a' and 's'. Hence, only 's'-terms for an arbitrary n will be considered without loss of generality.

The displacements $\mathbf{u}_n^s$ depend on the radial coordinate r and the vertical coordinate z. They can be written as

$$\mathbf{u}_n^s = \mathbf{H}(r)\cdot \mathbf{f}(z) \tag{2}$$

where the matrix $\mathbf{H}(r)$ is that part of the solution of the pertinent equations of motion, which depends upon r, whereas the functions $\mathbf{f}(z) = \{f_r(z), f_z(z), f_\varphi(z)\}^T$ are approximated by piecewise linear or quadratic shape functions.

To obtain $\mathbf{H}(r)$, the equations of motion are considered. In the transversely isotropic continuum they can be written for an arbitrary Fourier coefficient as

$$\begin{aligned}
&D_{rr}\cdot\frac{\partial \Delta_1}{\partial r} + (D_{rz}+2\cdot G_{rz})\cdot\frac{\partial \Delta_2}{\partial r} + (D_{rr}-D_{r\varphi})\cdot\frac{n}{r}\omega_z + 2\cdot G_{rz}\cdot\frac{\partial \omega_\varphi}{\partial z} + \rho\Omega^2\cdot u_n^s = 0\\
&D_{zz}\cdot\frac{\partial \Delta_2}{\partial z} + (D_{rz}+2\cdot G_{rz})\cdot\frac{\partial \Delta_1}{\partial z} - G_{rz}\cdot\frac{2}{r}\frac{\partial (r\cdot\omega_\varphi)}{\partial r} - 2\cdot G_{rz}\cdot\frac{n\cdot\omega_r}{r} + \rho\Omega^2\cdot w_n^s = 0\\
&D_{rr}\cdot\frac{n\Delta_1}{r} + (D_{rz}+2\cdot G_{rz})\cdot\frac{n\cdot\Delta_2}{r} + (D_{rr}-D_{r\varphi})\cdot\frac{\partial \omega_z}{\partial r} - 2\cdot G_{rz}\cdot\frac{\partial \omega_r}{\partial z} + \rho\Omega^2\cdot v_n^s = 0
\end{aligned} \tag{3}$$

with

$$\Delta_1 = \frac{u_n^s}{r} + \frac{\partial u_n^s}{\partial r} - n\cdot\frac{v_n^s}{r} \tag{3a}$$

$$\Delta_2 = \frac{\partial w_n^s}{\partial z} \tag{3b}$$

$$\omega_r = \frac{1}{2}\left(\frac{n}{r}\cdot w_n^s - \frac{\partial v_n^s}{\partial z}\right) \tag{3c}$$

$$\omega_\varphi = \frac{1}{2}\left(\frac{\partial u_n^s}{\partial z} - \frac{\partial w_n^s}{\partial r}\right) \tag{3d}$$

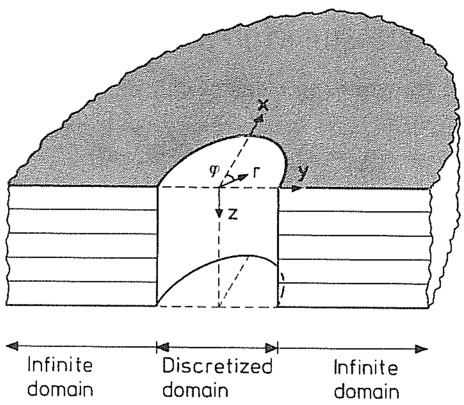

Figure 1. Transmitting element

$$\omega_z = \frac{1}{2}\left(\frac{\partial(nv_n^s)}{\partial r} - n\cdot u_n^s\right)\cdot\frac{1}{r} \tag{3e}$$

Herein Ω denotes the circular frequency of vibration and D_{rr}, D_{rz}, D_{zz}, $D_{r\varphi}$, G_{rz} are the five material constants of the transversely isotropic continuum, defined in the Appendix.

The solutions of the equations of motion for the isotropic medium using the principle of virtual displacements and the above displacement shape functions has been given by Kausel[8] and earlier for the special case $n = 0$ by Waas.[7] For a cross anisotropic medium a solution is given in Reference 6.

The fulfilment of the boundary conditions at the surface and at the base of the layered medium (zero stress and zero displacement, respectively) leads to two algebraic eigenvalue problems

$$\left(\begin{bmatrix}\mathbf{A}_{rr} & \mathbf{O}\\ \mathbf{O} & \mathbf{A}_{zz}\end{bmatrix}\cdot k_{R,v}^2+\begin{bmatrix}\mathbf{O} & \mathbf{B}_{rz}\\ \mathbf{B}_{rz}^T & \mathbf{O}\end{bmatrix}\cdot k_{R,v}+\begin{bmatrix}\mathbf{G}_{rr} & \mathbf{O}\\ \mathbf{O} & \mathbf{G}_{zz}\end{bmatrix}-\Omega^2\cdot\begin{bmatrix}\mathbf{M}_{rr} & \mathbf{O}\\ \mathbf{O} & \mathbf{M}_{zz}\end{bmatrix}\right)\cdot\begin{Bmatrix}\mathbf{f}_{r,v}\\ \mathbf{f}_{z,v}\end{Bmatrix} = \mathbf{O} \tag{4a}$$

$$(\mathbf{A}_{\varphi\varphi}\cdot k_{L,v}^2 + \mathbf{G}_{\varphi\varphi} - \Omega^2\cdot\mathbf{M}_{\varphi\varphi})\cdot\mathbf{f}_{\varphi,v} = \mathbf{O} \tag{4b}$$

The eigenvalues $k_{R,v}$ and $k_{L,v}$ correspond to the wave numbers (i.e. the reciprocal of the wavelength, multiplied by 2π) of generalized Rayleigh- and Love-waves, respectively. The vectors $\{\mathbf{f}_{r,v}\ \mathbf{f}_{z,v}\}^T$, $\mathbf{f}_{\varphi,v}$ describe the corresponding mode shapes. They contain the values of the functions $f_r(z)$, $f_z(z)$, $f_\varphi(z)$ at the layer interfaces in the vth mode. The matrices in equations (4a,b) are formed from the corresponding element matrices given in the Appendix using standard finite element assembly procedures.

The eigenvalue problems yield $4{*}n_L$ eigenvalues $k_{R,v}$ and $2{*}n_L$ eigenvalues $k_{L,v}$, where n_L is the total number of layers. For soil with material damping all eigenvalues are complex. Only those $3{*}n_L$ eigenvalues with a negative imaginary part are considered here in order to satisfy Sommerfeld's radiation principle.[7,8]

A complete solution for the displacement field is the weighted sum of the $3{*}n_L$ modes, or with equation (2) and the weighting factors α_v:

$$\mathbf{u}_n^s = \sum_{v=1}^{3*n_L} \mathbf{H}_v(r)\cdot f_v(z)\cdot\alpha_v \tag{5}$$

The matrix $\mathbf{H}(r)$ which is an analytical solution of equation (3), can be written as

$$\mathbf{H}_v(r) = \begin{bmatrix} \frac{1}{k_{Rv}}\cdot\frac{\partial H_n(k_{Rv}\cdot r)}{\partial r} & 0 & \frac{1}{k_{Lv}}\cdot\frac{n}{r}\cdot H_n(k_{Lv}\cdot r)\\ 0 & H_n(k_{Rv}r) & 0\\ \frac{1}{k_{Rv}}\cdot\frac{n}{r}\cdot H_n(k_{Rv}\cdot r) & 0 & \frac{1}{k_{Lv}}\cdot\frac{\partial H_n(k_{Lv}\cdot r)}{\partial r}\end{bmatrix} \tag{6}$$

where the functions $H_n(\ldots)$ are the Hankel functions of the second kind and of order n. At the boundary with radius $r = r_0$ the nodal displacements $\mathbf{u}_{n,0}^s$ are

$$\mathbf{u}_{n,0}^s = \mathbf{W}\cdot\alpha \tag{7}$$

with

$$\mathbf{u}_{n,0}^s = \begin{Bmatrix}\mathbf{u}_0^s\\ \mathbf{w}_0^s\\ \mathbf{v}_0^s\end{Bmatrix} \qquad \mathbf{W} = \begin{bmatrix}\mathbf{F}_r\cdot\mathbf{H}_R'\cdot\mathbf{K}_R'^{-1} & \frac{n}{r_0}\mathbf{F}_\varphi\cdot\mathbf{H}_L\cdot\mathbf{K}_L^{-1}\\ \mathbf{F}_z\cdot\mathbf{H}_R & \mathbf{O}\\ \frac{n}{r_0}\mathbf{F}_r\cdot\mathbf{H}_R\cdot\mathbf{K}_R^{-1} & \mathbf{F}_\varphi\cdot\mathbf{H}_L'\cdot\mathbf{K}_L^{-1}\end{bmatrix} \qquad \alpha = \begin{Bmatrix}\alpha_R\\ \alpha_L\end{Bmatrix} \tag{7a,b,c}$$

$$\mathbf{K}_R = \mathrm{Diag}\,(k_{R,v}) \tag{7d}$$

$$\mathbf{K}_L = \mathrm{Diag}\,(k_{L,v}) \tag{7e}$$

$$\mathbf{H}_R = \mathrm{Diag}\,(H_n(k_{R,v}\cdot r_0)) \tag{7f}$$

$$\mathbf{H}_L = \mathrm{Diag}\,(H_n(k_{L,v} \cdot r_0)) \tag{7g}$$

$$\mathbf{H}'_R = \mathrm{Diag}\left(\frac{\partial H_n(k_{R,v} \cdot r)}{\partial r}\bigg|_{r=r_0}\right) \tag{7h}$$

$\mathbf{u}_0^s$, $\mathbf{w}_0^s$, $\mathbf{v}_0^s$ are the vectors of the nodal displacements in the $r-$, $z-$, $\varphi-$direction, $\mathbf{F}_r$, $\mathbf{F}_z$, $\mathbf{F}_\varphi$ are the modal matrices of the eigenvectors $\mathbf{f}_{r,v}$, $\mathbf{f}_{z,v}$, $\mathbf{f}_{\varphi,v}$ respectively. The vectors α_R, α_L contain the mode participation factors α_v for Rayleigh- and Love-waves. The inverse relationship may be used to obtain the participation factors from given displacements $\mathbf{u}_{n,0}^s$ as

$$\alpha = \mathbf{W}^{-1} \cdot \mathbf{u}_{n,0}^s \tag{8}$$

Now the inhomogeneous problem of a layered continuum with a cylindrical boundary on which external forces are acting is considered. With the principle of virtual displacements, the nodal forces can be expressed in terms of the nodal displacements at the boundary, resulting in

$$\mathbf{P}_{n,0}^s = \mathbf{R}_n \cdot \mathbf{u}_{n,0}^s \tag{9}$$

with

$$\mathbf{P}_{n,0}^s = \{\mathbf{P}_r\ \mathbf{P}_z\ \mathbf{P}_\varphi\}^T \tag{9a}$$

The vectors $\mathbf{P}_r$, $\mathbf{P}_z$, $\mathbf{P}_\varphi$ contain the forces per radian in the radial, vertical and tangential direction, respectively.

The stiffness matrix of the transmitting element is obtained as

$$\mathbf{R}_n = \mathbf{Q} \cdot \mathbf{W}^{-1} \tag{10}$$

with

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} \end{bmatrix} \tag{10a}$$

$$\mathbf{Q}_{11} = r_0 \cdot \mathbf{A}_{rr} \cdot \mathbf{F}_r \cdot \mathbf{K}_R \cdot \mathbf{H}_R + 2 \cdot \mathbf{A}_{\varphi\varphi} \cdot \mathbf{F}_r \cdot \mathbf{K}_R^{-1} \cdot \left(-\frac{n^2}{r_0}\mathbf{H}_R + \mathbf{H}'_R\right) + r_0 \cdot \mathbf{D}_{rz} \cdot \mathbf{F}_z \cdot \mathbf{H}_R, \tag{10b}$$

$$\mathbf{Q}_{12} = -2 \cdot n \cdot \mathbf{A}_{\varphi\varphi} \mathbf{F}_\varphi \cdot \mathbf{K}_L^{-1} \cdot \left(\mathbf{H}'_L - \frac{1}{r_0} \cdot \mathbf{H}_L\right) \tag{10c}$$

$$\mathbf{Q}_{21} = r_0 \cdot \mathbf{D}_{zr} \mathbf{F}_r \cdot \mathbf{H}'_R \cdot \mathbf{K}_R^{-1} - r_0 \cdot \mathbf{A}_{zz} \cdot \mathbf{F}_z \cdot \mathbf{H}'_R \tag{10d}$$

$$\mathbf{Q}_{22} = \mathbf{n} \cdot \mathbf{D}_{zr} \cdot \mathbf{F}_\varphi \cdot \mathbf{K}_L^{-1} \cdot \mathbf{H}_L \tag{10e}$$

$$\mathbf{Q}_{31} = -2 \cdot n \cdot \mathbf{A}_{\varphi\varphi} \cdot \mathbf{F}_r \cdot \mathbf{K}_L^{-1} \cdot \left(\mathbf{H}'_R - \frac{1}{r_0} \cdot \mathbf{H}_R\right) \tag{10f}$$

$$\mathbf{Q}_{32} = \mathbf{A}_{\varphi\varphi} \cdot \mathbf{F}_\varphi \cdot \mathbf{K}_L^{-1} \cdot \left(2 \cdot \mathbf{H}'_L + r_0 \cdot \left(\mathbf{K}_L^2 - 2\frac{n^2}{r_0^2}\right) \cdot \mathbf{H}_L\right) \tag{10g}$$

The stiffness matrix may be used directly with axisymmetric elements in the discretized domain. For a non-axisymmetric discretization using three-dimensional isoparametric solid elements within the cylindrical boundary, a transformation of the stiffness matrix has to be performed.[10]

EXAMPLE

A viscoelastic homogeneous layer with a vertical surface load is considered. The uniformly distributed load P acts on a square area of $2b * 2b$ and is harmonically varying in time, Figure 2.

A cross anisotropic as well as an isotropic medium are investigated. The material properties are given in Table I. In the cross anisotropic medium the ratio of the moduli of elasticity in horizontal and in vertical

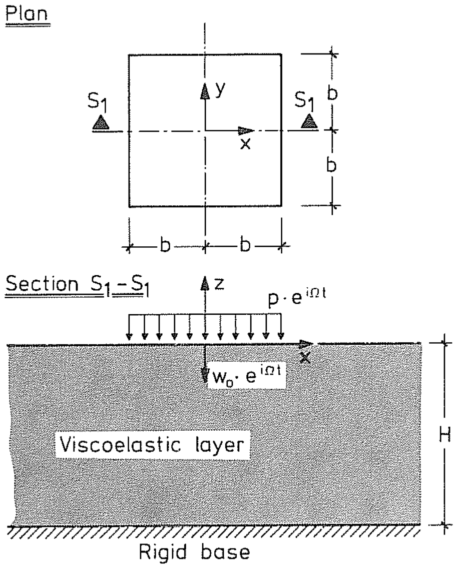

Figure 2. Soil model

direction is $E_H/E_V = 3$ (or $D_{rr}/D_{zz} = 2{\cdot}5$). However, in both media the shear moduli G_{rz} in a vertical plane and the vertical stiffness D_{zz} are identical.

The damping in the medium is assumed to be of viscous type. If for instance $\tilde{G}_{rz}$ denotes the real shear modulus in a vertical plane, the complex shear modulus G_{rz} can be written

$$G_{rz} = \tilde{G}_{rz}\,(1 + \mathrm{i}\cdot a_0 \cdot \eta) \tag{11}$$

with the dimensionless frequency

$$a_0 = \frac{\Omega \cdot b}{\sqrt{(\tilde{G}_{rz}/\rho)}} \tag{11a}$$

The computation has been performed for a ratio $H/b = 2$ and a damping coefficient $\eta = 0{\cdot}5$. The discretisation of one eighth of the system with three-dimensional isoparametric solid elements with quadratic shape functions is shown in Figure 3. Because of the non-axisymmetry of the model in the discretized domain the stiffness matrix of the transmitting boundary is transformed according to References 10 and 11 using three Fourier terms ($n = 0, 4, 8$).

The results are given in Figure 4 as compliance functions for the central point of the square area loaded. The functions f_{1v} and f_{2v} are defined as

$$w_0 = \frac{4 \cdot \mathrm{b}}{\tilde{G}_{rz}} \cdot (f_{1v} - \mathrm{i} \cdot f_{2v}) \cdot p \tag{12}$$

They depend on the dimensionless frequency a_0.

The compliance functions of the isotropic medium compare well with those of an analytical solution, given in Reference 12. The solution for the anisotropic medium differs from the solution of the isotropic medium by

Table I. Material properties

	D_{rr}	D_{rz}	D_{zz}	$G_{r\varphi}$	G_{rz}
Isotropic	3·0	1·0	3·0	1·0	1·0
Anisotropic	7·47	2·57	3·0	2·3	1·0

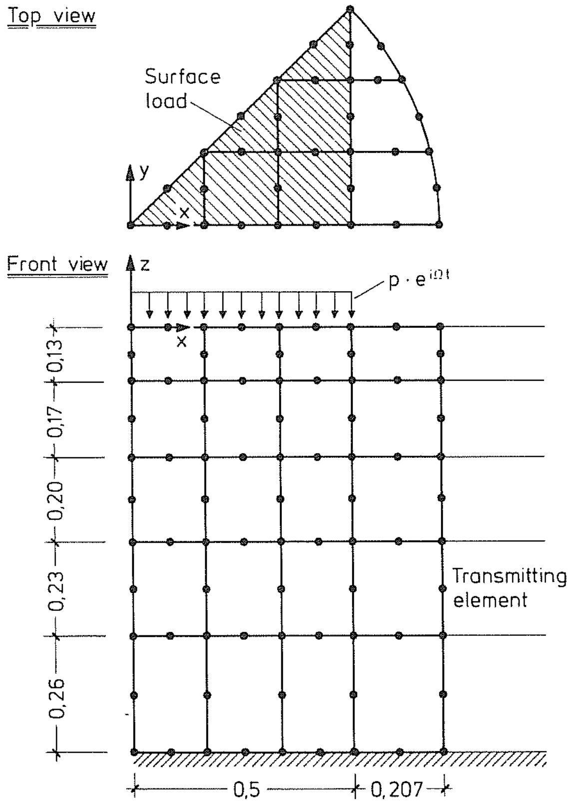

Figure 3. Finite element model

about 10 to 15 per cent. This applies to the static as well as to the dynamic displacements. The example indicates that some cross anisotropic media may be approximated by an isotropic medium with adequate accuracy.

APPENDIX: MATERIAL LAW AND LAYER MATRICES

Material law

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\varphi\varphi} \\ \sigma_{zz} \\ \tau_{rz} \\ \tau_{r\varphi} \\ \tau_{\varphi z} \end{Bmatrix} = \begin{bmatrix} D_{rr} & D_{r\varphi} & D_{rz} & & & \\ D_{r\varphi} & D_{rr} & D_{rz} & & \mathbf{O} & \\ D_{rz} & D_{rz} & D_{zz} & & & \\ & & & G_{rz} & & \\ & \mathbf{O} & & & G_{r\varphi} & \\ & & & & & G_{rz} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{zz} \\ \gamma_{rz} \\ \gamma_{r\varphi} \\ \gamma_{\varphi z} \end{Bmatrix} \tag{A1}$$

$$G_{r\varphi} = \tfrac{1}{2} \cdot (D_{rr} - D_{r\varphi})$$

All elements of equation (A1) are complex in order to include the material damping of the medium.

Layer matrices

Material constants and layer thickness h refer to layer 1. The first and the second indices of the following layer stiffness matrices refer to the direction of the forces and of the displacements, respectively.

$$\mathbf{A}_{rr} = D_{rr} \cdot h \cdot \mathbf{I}_a \tag{A2}$$

$$\mathbf{A}_{\varphi\varphi} = \tfrac{1}{2}(D_{rr} - D_{r\varphi}) \cdot h \cdot \mathbf{I}_a \tag{A3}$$

$$\mathbf{A}_{zz} = G_{rz} \cdot h \cdot \mathbf{I}_a \tag{A4}$$

$$\mathbf{B}_{rz} = D_{rz} \cdot \mathbf{I}_b - G_{rz} \cdot \mathbf{I}_b^T \tag{A5}$$

$$\mathbf{G}_{rr} = G_{rz} \cdot \frac{1}{h} \cdot \mathbf{I}_c \tag{A6}$$

$$\mathbf{G}_{zz} = D_{zz} \cdot \frac{1}{h} \cdot \mathbf{I}_c \tag{A7}$$

$$\mathbf{G}_{\varphi\varphi} = \mathbf{G}_{rr} \tag{A8}$$

$$\mathbf{M}_{rr} = \rho \cdot h \cdot \mathbf{I}_a \tag{A9}$$

$$\mathbf{M}_{zz} = \mathbf{M}_{\varphi\varphi} = \mathbf{M}_{rr} \tag{A10}$$

$$\mathbf{D}_{rz} = D_{rz} \cdot \mathbf{I}_b \tag{A11}$$

$$\mathbf{D}_{zr} = G_{rz} \cdot \mathbf{I}_b \tag{A12}$$

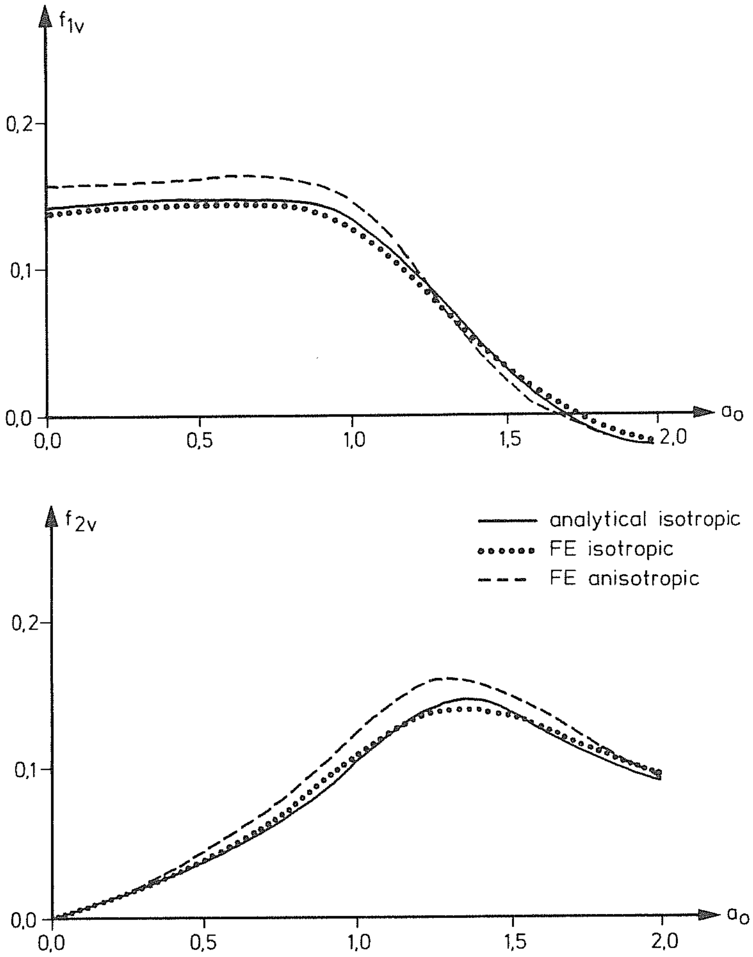

Figure 4. Compliance functions

For quadratic expansion

$$\mathbf{I}_a = \frac{1}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \tag{A13}$$

$$\mathbf{I}_b = \frac{1}{6}\begin{bmatrix} 3 & -4 & 1 \\ 4 & 0 & -4 \\ -1 & 4 & -3 \end{bmatrix} \tag{A14}$$

$$\mathbf{I}_c = \frac{1}{3}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \tag{A15}$$

For linear expansion the matrices are transformed as

$$\tilde{\mathbf{I}}_a = \lambda^{\mathrm{T}} \cdot \mathbf{I}_a \cdot \lambda \qquad \text{with} \qquad \lambda = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \tag{A16}$$

$$\tilde{\mathbf{I}}_b = \lambda^{\mathrm{T}} \mathbf{I}_b \lambda$$

$$\tilde{\mathbf{I}}_c = \lambda^{\mathrm{T}} \mathbf{I}_c \lambda$$

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