Analysis of flat slabs by the finite element method[1)]

Horst Werkle
Department of Civil Engineering, University of Applied Sciences Konstanz
D-78476 Konstanz, Germany

ABSTRACT

This paper presents a consistent approach for modeling the connection between the column and the slab in the finite element analysis of concrete flat slabs. The model takes into account the normal stiffness as well as the bending stiffness of the column. The stress resultants of the column are transformed into the nodal forces of the finite element model of the plate. Similarly, the transformation for the displacements is formulated. Using both relationships, the stiffness matrix of the column can be transformed to the nodal points of the plate. This is denoted as equivalent stiffness transformation (EST). The model also gives the normal force and the bending moments in the column, to be used in the column design. The model is particularly useful for taking into account the bending effects in edge and corner columns of flat slabs.

1 Introduction

Concrete flat slabs are widely used in central Europe and other parts of the world where concrete rather than structural steel is the preferred building medium. They are made from a reinforced or prestressed concrete slab supported by columns.

The finite element method is well suited for the analysis of flat slabs. However, the modelling of the connection of the column and the slab is not evident. Normally, the column is considered as beam and the slab as plate structure. The problem in modelling the transition between beam and plate structures lies in the different descriptions of the stresses and sectional forces in both types of structures. The concept of point forces and moments which is successful in the analysis of beams results in singularities when applied to plates. Therefore, point supports are not well suited to model columns in the analysis of flat slabs. Different models are currently in use. All of them show some inconsistencies. In this paper a new improved model based on the concept of distributed supports is presented.

---

1) presented at ASEM'02, Busan, Korea, 2002

2 State of the art

Flat slabs have been analysed for many years by means of the finite element method. The columns were first modelled as point supports, i.e. a nodal point of the finite elements model of the plate has been considered to be vertically restrained. Because of the restrictions and deficiencies of this simplified approach, improved models were later developed. The following are those most commonly used today, Fig. 1:

- Point support
- Elastic support (Winkler springs)
- Fluid cushion model (constant pressure by the column head)
- Rigid column head model

Three-dimensional solid models for the column head used in research are generally not considered to be appropriate for practical application since they are not consistent with the design model of the column and the plate.

Point support

A point support introduces a point force into the plate. This results in a singularity of bending moments and shear forces in the plate at the point of application. In practice the model is used for slender columns (Kemmler, Ramm, 2001).

Elastic support

An improved model is the elastic support of the plate in the area of the column cross section using Winkler springs. This model avoids any singularity of sectional forces. The spring stiffness is usually determined based on the normal stiffness of the column. This assumption gives the spring stiffness as $k_z = E \cdot A_s / h$ and the Winkler modulus as $k_{s\_z} = E/h$ where A_s denotes the cross section area, E the Young's modulus and h the height of the column. However, this stiffness does not represent the bending stiffness of the column. The rotational spring constant of the column is $k_{\phi y} = \alpha \cdot E \cdot I_y / h$ where I_y is the moment of inertia of the column cross section and α a factor depending on the support condition of the opposite end of the column. For pin support $\alpha = 3$ and for clamped support $\alpha = 4$ are obtained. In case of two columns, one of the upper and one of the lower story, the stiffnesses are added, e.g. for identical columns with clamped ends $\alpha = 8$. The Winkler spring modulus corresponding to the rotational spring is obtained as $k_{s\_z} = \alpha \cdot E/h_s$. Both modes – vertical displacement and rotation - cannot be described by a single Winkler modulus.

Fluid cushion model

The fluid cushion model is based on the assumption that stresses applied by the column on the plate are uniformly distributed. This assumption may be appropriate for inner panels of slabs where bending effects of columns are not predominant. It is not suited for columns in edge or corner panels since the effect of bending of the columns is neglected.

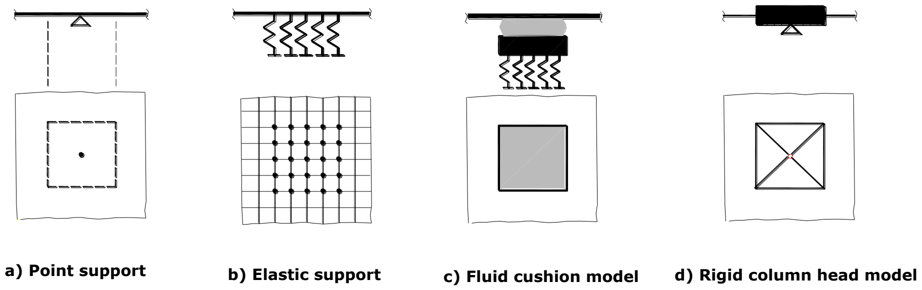


Fig. 1. Models for columns in the analysis of flat slabs

Rigid column head model

The head of the column is assumed to be rigid (Hartmann, Katz, 2002). The conditions of rigidity can be defined as rigid links and the corresponding degrees of freedom can be eliminated. In this way numerical problems in the solution of the system of equations are avoided. Normally, this model gives no results in the plate over the column head. The inclusion of a rigid part in an elastic structure may be the origin of stress singularities near the rigid structural part. It may also distort the sectional forces in the plate near the column head considerably.

In order to avoid rigid inclusions in the finite element model, the elastic support model is often preferred. However, a consistent representation of the normal and the bending stiffness of the column is not possible with this model. Therefore, a new model has been developed in order to represent the normal and bending stiffness of a column consistently.

3 EST model

The basic assumption of the equivalent stiffness transformation model (EST model) is the linear distribution of the longitudinal stresses according to the beam theory in the column. They can be written

$$p(\xi,\eta) = \frac{F_z}{A_z} - \frac{M_\eta}{I_\eta}\cdot\xi + \frac{M_\xi}{I_\xi}\cdot\eta \qquad \text{or} \qquad p(\xi,\eta) = \underline{X}_{\xi,\eta}\cdot\underline{I}\cdot\underline{F}_{St}$$

$$\text{with}\quad \underline{X}_{\xi,\eta} = \begin{bmatrix}1 & \xi & \eta\end{bmatrix}, \qquad \underline{I} = \begin{bmatrix} 1/A_z & 0 & 0 \\ 0 & -1/I_\eta & 0 \\ 0 & 0 & 1/I_\xi \end{bmatrix}, \qquad \underline{F}_{St} = \begin{bmatrix} F_z \\ M_\eta \\ M_\xi \end{bmatrix}. \tag{1}$$

They are related to the principal axes ξ, η of the column cross section, Fig. 2. The stresses are now applied as "distributed loads" to the finite elements of the slab. Over the column cross section the plate is arbitrarily discretizised into quadrilateral finite elements. The stresses applied from the column to the plate elements are transformed to equivalent nodal forces using the principle of virtual displacements. For a single 4-node element one obtains the nodal forces $\underline{F}^{(el)}$ as:

$$\underline{F}^{(el)} = \int \underline{N} \cdot p \cdot dA \quad \text{with} \quad \underline{F}^{(el)} = \begin{bmatrix} F_{z1} \\ F_{z2} \\ F_{z3} \\ F_{z4} \end{bmatrix}, \quad \underline{N}(r,s) = \frac{1}{4} \cdot \begin{bmatrix} (1+r)\cdot(1+s) \\ (1-r)\cdot(1+s) \\ (1-r)\cdot(1-s) \\ (1+r)\cdot(1-s) \end{bmatrix}, \quad dA = Det(\underline{J}) \cdot dr \cdot ds\,. \quad (2)$$

The functional determinant $Det(\underline{J})$ is the determinant of the Jacobi operator. With x_i, y_i as coordinates of the nodes of the finite element one obtains:

$$\underline{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \quad \text{with} \quad x(r,s) = \underline{N}(r,s)^T \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{and} \quad y(r,s) = \underline{N}(r,s)^T \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}. \quad (2)$$

The stresses p, applied at the plate elements can be interpolated from their nodal values p_i as

$$p(r,s) = \underline{N}(r,s)^T \cdot \underline{p}^{(el)} \quad \text{with} \quad \underline{p}^{(el)^T} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix} \quad (3)$$

Now the nodal forces are written with (1), (2) and (3):

$$\underline{F}^{(el)} = \underline{A}^{(el)} \cdot \underline{p}^{(el)} \quad \text{with} \quad \underline{A}^{(el)} = \iint \underline{N} \cdot \underline{N}^T \cdot Det(\underline{J})\; dr\; ds \quad (4)$$

The integral can be computed numerically by Gauss integration. However, for the quadrilateral element it also can be solved analytically (Werkle, 2001). For a rectangular element with the side lengths a and b in x- and y-direction respectively, one obtains:

$$\underline{A}^{(el)} = \frac{a \cdot b}{36} \begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} \quad (5)$$

The element forces $\underline{F}^{(el)}$ and the applied stresses $\underline{p}^{(el)}$ of a single element are now related to the nodal points of the finite elements at the column head by a topology matrix. One obtains:

$$\underline{F}_{Pl}^{(el)} = \underline{Z}^{(el)^T} \cdot \underline{F}^{(el)} \quad \text{and} \quad \underline{p}^{(el)} = \underline{Z}^{(el)} \cdot \underline{p}_{Pl} \quad (6)$$

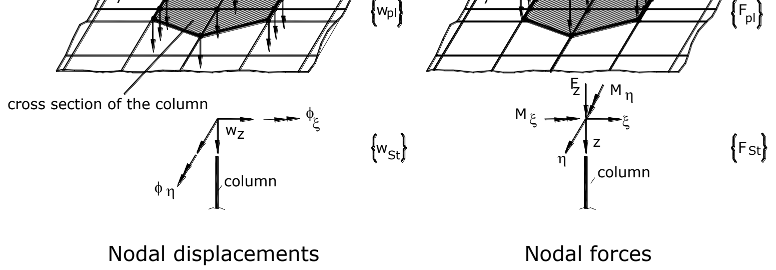


Fig. 2. EST model

With these relationships the nodal forces and the distributed loads can be related to the nodal points of the finite element model of the plate. Summing up the contributions for all elements over the column head one obtains

$$\underline{F}_{pl} = \underline{A} \cdot \underline{p}_{Pl} \qquad \text{with} \quad \underline{A} = \sum_{(el)} \underline{Z}^{(el)^T} \cdot \underline{A}^{(el)} \cdot \underline{Z}^{(el)} \tag{7}$$

Here $\underline{F}_{pl}$ and $\underline{p}_{Pl}$ denote the contact forces and stresses related to the finite elements of the EST element. The stresses at the nodal points of the finite elements are obtained by introducing the corresponding coordinates in Eq.(1) as

$$\underline{p}_{Pl} = \underline{\Xi} \cdot \underline{I} \cdot \underline{F}_{St} \tag{8}$$

with $\underline{p}_{Pl} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ . \\ . \\ p_n \end{bmatrix}$ and $\underline{\Xi} = \begin{bmatrix} 1 & \xi_1 & \eta_1 \\ 1 & \xi_2 & \eta_2 \\ 1 & \xi_3 & \eta_3 \\ . & . & . \\ . & . & . \\ 1 & \xi_n & \eta_n \end{bmatrix}$.

With $F_{Pl} = \underline{A} \cdot \underline{p}_{Pl}$ according to Eq.(7) the nodal forces which are equivalent to the sectional forces $\underline{F}_{St}$ at the column head are now

$$\underline{F}_{Pl} = \underline{T}^T \cdot \underline{F}_{St} \tag{9}$$

with $\underline{T}^T = \underline{A} \cdot \underline{\Xi} \cdot \underline{I}$. (10)

Now the transformation of the displacements and the rotations at the column head into the coordinate system of the plate is considered. It can be shown that the same transformation is valid, i.e.

$$\underline{w}_{St} = \underline{T} \cdot \underline{w}_{Pl}\,, \tag{11}$$

where $\underline{w}_{Pl} = \begin{bmatrix} w_1 & w_2 & \ldots\ldots & w_n \end{bmatrix}^T$ (11a)

denotes the displacement of the nodal points of the plate and

$$\underline{w}_{St} = \begin{bmatrix} w_z & \phi_\eta & \phi_\xi \end{bmatrix}^T \tag{11b}$$

describes the vertical translation and the two rotations of the column head, Fig. 2. The displacements $\underline{w}_{St}$ according to Eq.(11) are a weighted average of the nodal displacements $\underline{w}_{Pl}$.

The stiffness of the column is given by

$$\begin{bmatrix} F_z \\ M_\eta \\ M_\xi \end{bmatrix} = \begin{bmatrix} k_z & 0 & 0 \\ 0 & k_{\eta\eta} & 0 \\ 0 & 0 & k_{\xi\xi} \end{bmatrix} \cdot \begin{bmatrix} w_z \\ \phi_\eta \\ \phi_\xi \end{bmatrix} \tag{12}$$

$$\underline{F}_{St} = \underline{K}_{St} \cdot \underline{w}_{St} \tag{12a}$$

where the spring constant k_z relates to the vertical displacement, $k_{\eta\eta} = \alpha \cdot E \cdot I_\eta / h$ to the rotation about the η- axis and $k_{\xi\xi} = \alpha \cdot E \cdot I_\xi / h$ to the rotation about the ξ-axis.

The column stiffness according to Eq.(12) can now be transformed onto the nodal points of the plate. Using Eq.(9) and Eq.(10) one obtains

$$\underline{F}_{Pl} = \underline{K}_{Pl} \cdot \underline{w}_{Pl}, \qquad (13)$$

where

$$\underline{K}_{Pl} = \underline{T}^T \cdot \underline{K}_{St} \cdot \underline{T} \qquad (14)$$

represents the stiffness matrix of the EST element.

After the solution of the system equations of the finite element model the element stresses and sectional forces are computed. For the EST element the forces at the column head are

$$\underline{F}_{St} = \underline{K}_{St} \cdot \underline{w}_{St} = \underline{K}_{St} \cdot \underline{T} \cdot \underline{w}_{Pl} \qquad (15)$$

The normal force and the bending moments at the column head can be used for the design of the column.

4 EST element for a rectangular column

The plate is discretizised over the column cross section in 4 elements, Fig. 3 (Werkle 2000). The transformation matrix $\underline{T}$ according to Eq.(10) is evaluated as

$$\underline{T} = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{4 \cdot d_x} & 0 & \frac{-1}{4 \cdot d_x} & \frac{1}{2 \cdot d_x} & 0 & \frac{-1}{2 \cdot d_x} & \frac{1}{4 \cdot d_x} & 0 & \frac{-1}{4 \cdot d_x} \\ \frac{1}{4 \cdot d_y} & \frac{1}{2 \cdot d_y} & \frac{1}{4 \cdot d_y} & 0 & 0 & 0 & \frac{-1}{4 \cdot d_y} & \frac{-1}{2 \cdot d_y} & \frac{-1}{4 \cdot d_y} \end{bmatrix} \text{ with } \underline{w}_{St} = \begin{bmatrix} w_z \\ \phi_{yy} \\ \phi_{xx} \end{bmatrix}. \qquad (16)$$

Their columns relate to the vertical displacements of the nodal points

$$\underline{w}_{Pl} = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 \end{bmatrix}. \qquad (17)$$

The element stiffness matrix obtained by Eq.(15) with $\underline{T}$ acc. to Eq.(16) and $\xi \hat{=} x$, $\eta \hat{=} y$.

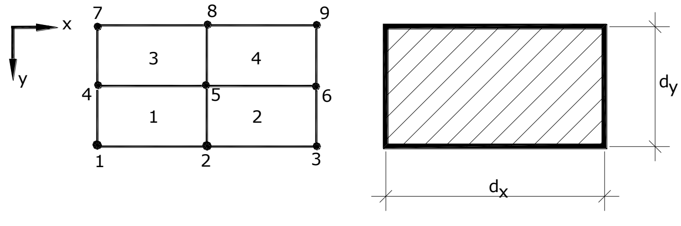


Fig. 3. Finite element assemblage for a rectangular column

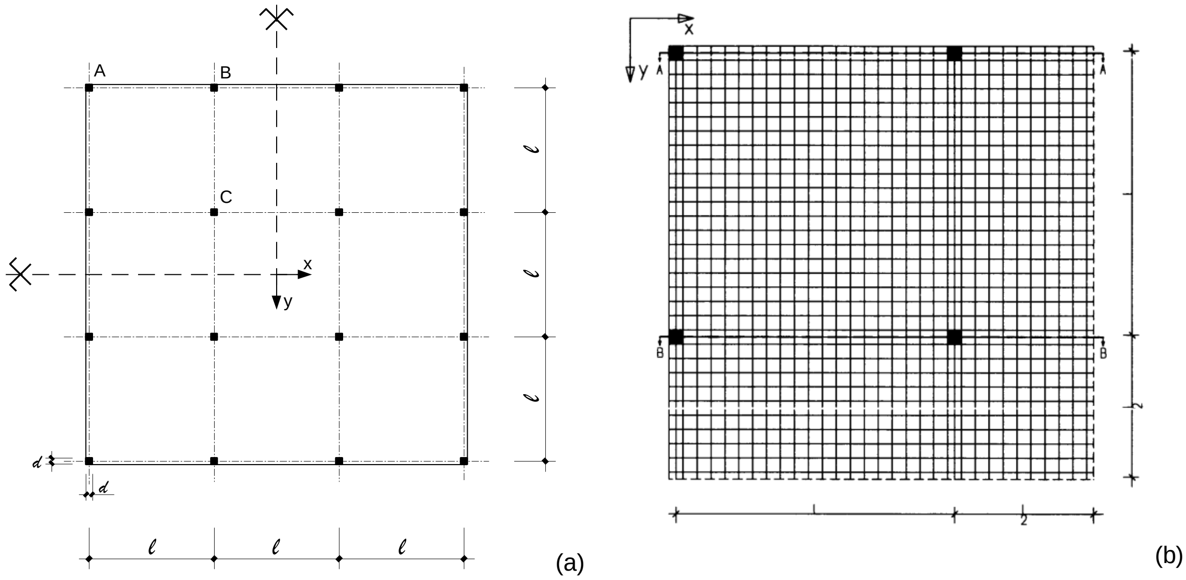


Fig. 4. Flat slab: (a) Flat with 3x3 panels; (b) Finite element model of a quarter of the slab

5 Example

The application of the EST element is shown for the slab with free edges and 3x3 panels, given in Fig. 3. The slab is loaded by a constant distributed load *p*, its thickness is $d_{Pl} = \ell / 30$, the Poisson ratio is $\mu = 0.2$. The quadratic columns with the dimensions $a = b = \ell / 20$ and the height $h_S = \ell / 2$ are pin-supported at the lower end. The analysis is done using the plate element in the program SEPP (Sofistik GmbH, Unterschleißheim, Germany) based on Mindlin's theory.

For comparison two models with an elastic support by Winkler springs are investigated. The Winkler moduli have been chosen as $k_{s\_z} = E/h_S$ corresponding to the normal stiffness of the column as well as $k_{s\_z} = 3 \cdot E/h_S$ corresponding to its bending stiffenss.

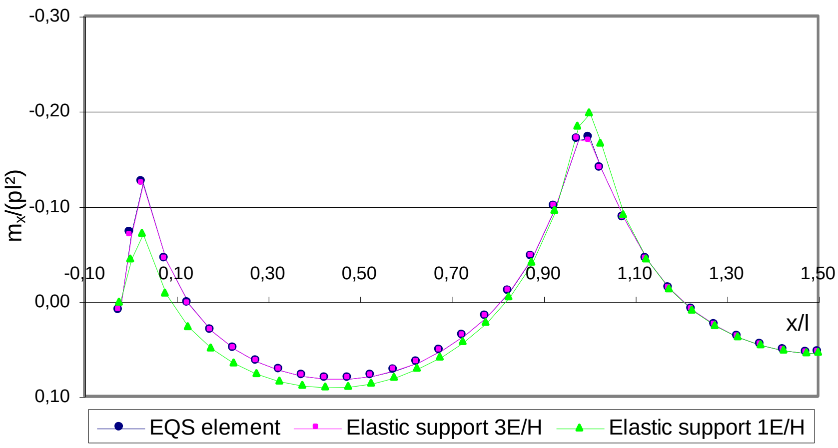
Figure 5 shows the bending moment m_x in sections A-A and B-B for the two Winkler spring constants and for the EST element. The sectional forces in the column are given in Table 1. In the corner column and in the edge column, the column stiffness influences the bending moments considerably. However, at the internal panel its influence can be neglected. The results of the elastic support with a Winkler modulus of $k_{s\_z} = 3 \cdot E/h_S$ agree well with the EST model in the example presented. If the EST model is not available in the program used, an elastic support with the Winkler modulus of the rotation should be used instead.

Column	$F_z /(p \cdot \ell^2)$	$M_x /(p \cdot \ell^3)$	$M_y /(p \cdot \ell^3)$
A	0.219	0.0134	-0.0134
B	0.474	0.0218	0.0003
C	1.157	-0.0004	0.0004

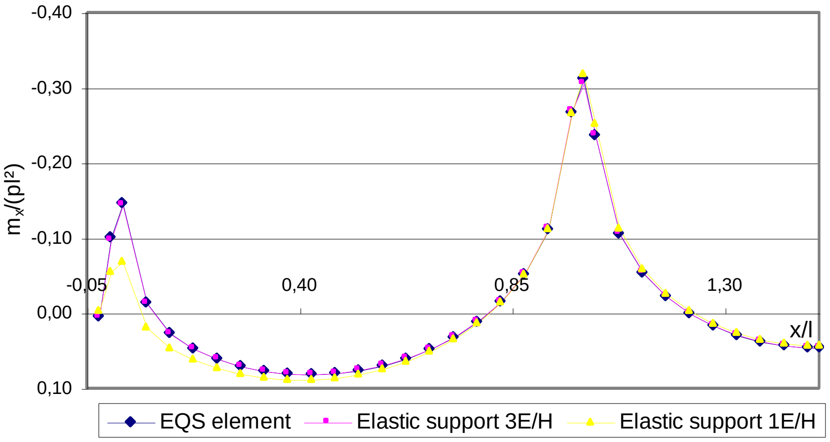
Table 1.
Stress resultants in columns

6 Conclusions

The EST model is well suited to represent the normal as well as the bending stiffness of columns with arbitrary cross section in the analysis of flat slabs. Since it shows no singularities it can also be used with adaptive mesh refinement. The concept of equivalent stress models can also be applied to other cases where the transition between finite elements with a different description of stresses has to be modelled.



(a) Section A-A



(b) Section B-B

Fig. 5. Bending moment m_x

REFERENCES

Hartmann F., Katz C. (2002), *Statik mit finiten Elementen*, Springer, Berlin, 2002

Kemmler R., E. Ramm (2001), „Modellierung mit der Methode der Finiten Elemente“, *Beton-Kalender 2001*, Ernst & Sohn, Berlin

Werkle H. (2000), Konsistente Modellierung von Stützen bei der Finite-Element-Berechnung von Flachdecken, *Bautechnik*, 77 (2000), Ernst & Sohn, Berlin, Germany, 416-425.

Werkle H. (2001), *Finite Elemente in der Baustatik*, 2nd edition, Vieweg, Wiesbaden, Germany