Finite Elements in Structural Analysis

Introduction

2 Truss and beam structures

Plate and shell structures

Modeling

Methods of structural analysis

Classical methods of structural analysis

- Force method
- Displacement method

Finite Element Method (FEM)

- The Finite Element Method is a generalisation of the displacement method for structural analysis in matrix notation.
- For truss and beam structures it is also denoted as the **Direct Stiffness Method** (DSM).

Introductory example: Truss system

Nodal points

here: nodal points 1- 4

Elements

here: truss elements 1-6

Degrees of freedom

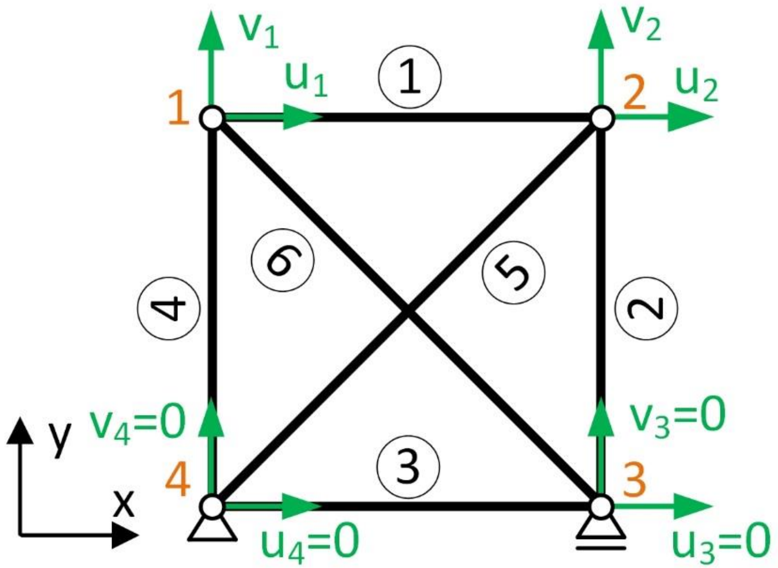
Degrees of freedom are independently movable displacements or rotations of nodal points.

here: u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4

Support conditions

Restraints of individual degrees of freedom

here: $v_3 = 0$, $u_4 = 0$, $v_4 = 0$



Introductory example: Truss system

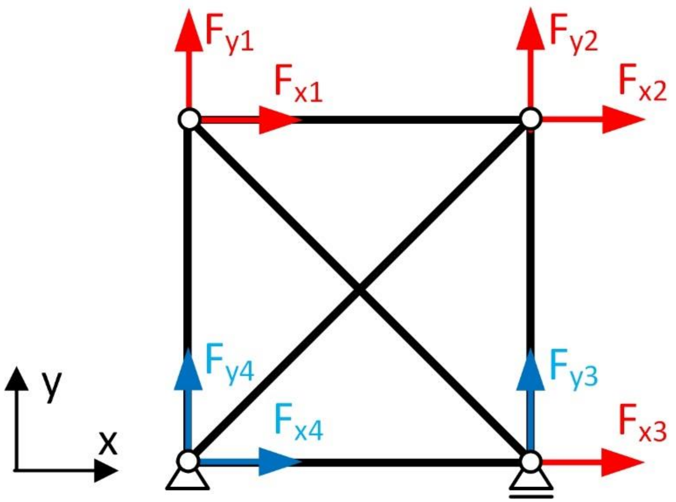
Nodal point forces

External forces

here: F_{x1},F_{y1},F_{x2}, F_{y2}, F_{x3}

Support forces

here: F_{y3},F_{x4},F_{y4}



Global coordinate system

Nodal point forces and nodal displacements are specifed in the global coordinate system.

here: x, y

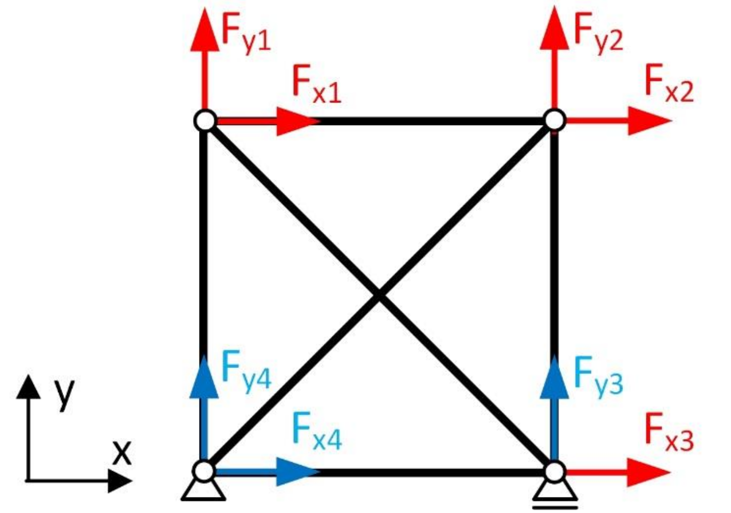
Sign rule

Nodal point forces and nodal displacements are positive in the direction of the positive coordinate axes of the global coordinate system.

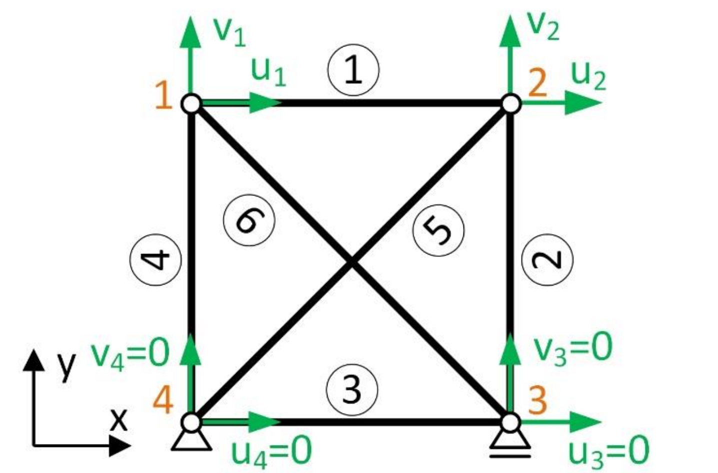
Introductory example: Truss system

System of equations

- The unknowns are displacements (displacements and rotations).
- The coefficient matrix is called the **global stiffness matrix of the system.**
- The right-hand side consists of the nodal point forces, i.e. the loads acting at the nodal points.



$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \end{bmatrix}$$



here: $v_3=u_4=v_4=0$ They are omitted in the matrix due to the support conditions

Global stiffness matrix

Characteristics of the system of equations

1. For stable i.e. not kinematic structural systems the system of equations has a unique solution. The global stiffness matrix is regular.
2. Diagonal terms are always positive (spring constants)
3. The stiffness matrix is symmetric
4. The global stiffness matrix is assembled from the stiffness matrices of the finite elements

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \end{bmatrix}$$

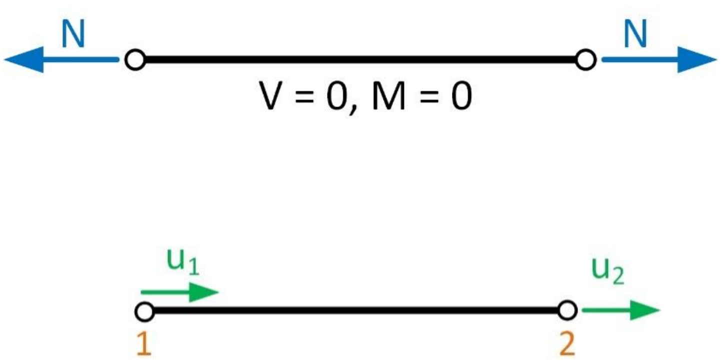
The solution of the system of equations gives the nodal point displacements.

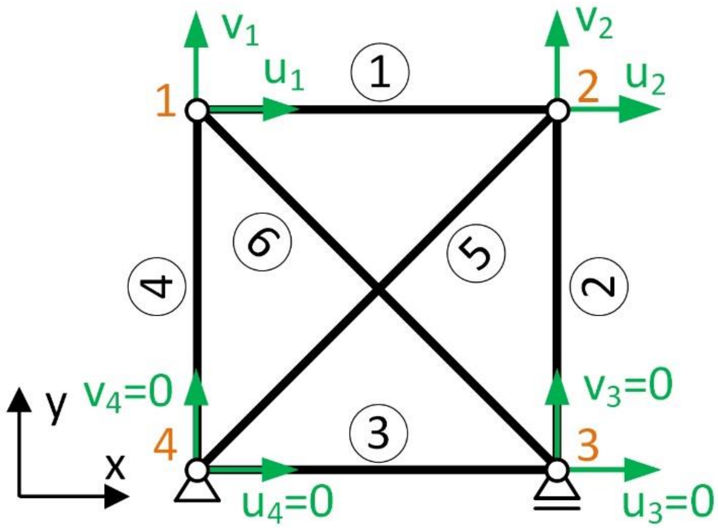
Global stiffness matrix

Section forces and element stresses

The section forces and element stresses are determined ***element by element*** using the nodal point displacements.

here: Normal forces and normal stresses in the truss elements





Computational steps of the Finite Element Method

1. Determination of the element stiffness matrices and the nodal point loads.
2. Assemblage of the global stiffness matrix with the element stiffness matrices and of the global load vector with the nodal loads.

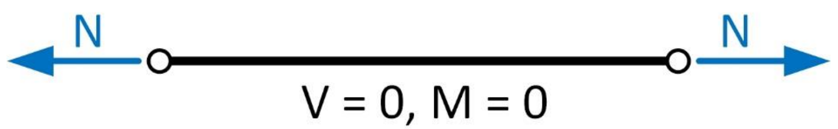
$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \end{bmatrix}$$

3. Solution of the system of equations with the global stiffness matrix gives the nodal point displacements.
4. Determination of the support reactions using the nodal point displacements.
5. Determination of the element stresses / section forces using the nodal point displacements.

Element stiffness matrix of a truss element

Stiffness matrix in local coordinates

Definition: A beam with normal forces only is called truss element



Section force

Nodal point displacements

Nodal point forces

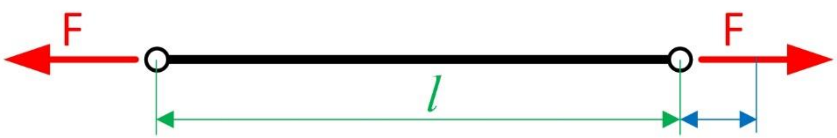
$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Element stiffness matrix

Element stiffness matrix of a truss element

Derivation of the stiffness matrix

Elongation of a truss element



$$\delta = \varepsilon \cdot \ell = \frac{\sigma}{E} \cdot \ell = \frac{F \cdot \ell}{E \cdot A}$$

A= Cross section area
E= Young´s modulus
Hooke‘s law:

$$\sigma = E \cdot \varepsilon \rightarrow \varepsilon = \sigma / E$$

Normal force

$$F = \frac{E \cdot A}{\ell} \cdot \delta \quad with \quad \delta = u_2 - u_1$$

Element forces

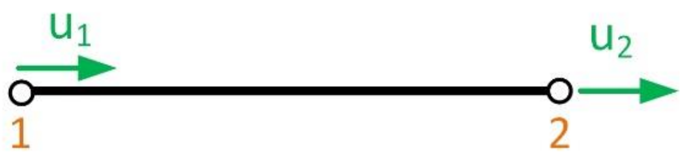
$$F_1 = -F = \frac{E \cdot A}{\ell} \left(u_1 - u_2 \right)$$

$$F_2 = F = \frac{E \cdot A}{\ell} \left(-u_1 + u_2 \right)$$

Element stiffness matrix of a truss element

Derivation of the stiffness matrix

Element forces



$$F_1 = \frac{E \cdot A}{\ell} \left(u_1 - u_2\right)$$

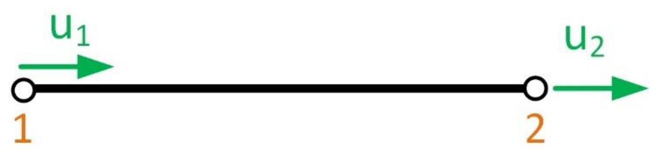
$$F_2 = \frac{E \cdot A}{\ell} \left(-u_1 + u_2\right)$$

In matrix notation: Element stiffness matrix

$$\frac{E \cdot A}{\ell} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Element stiffness matrix of a truss element

Element stiffness matrix of a truss element in local coordinates



$$\frac{E \cdot A}{\ell} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\underline{K}^{(lok)} \cdot \underline{u}^{(lok)} = \underline{F}^{(lok)}$$

$\underline{K}^{(lok)}$ = Element stiffness matrix

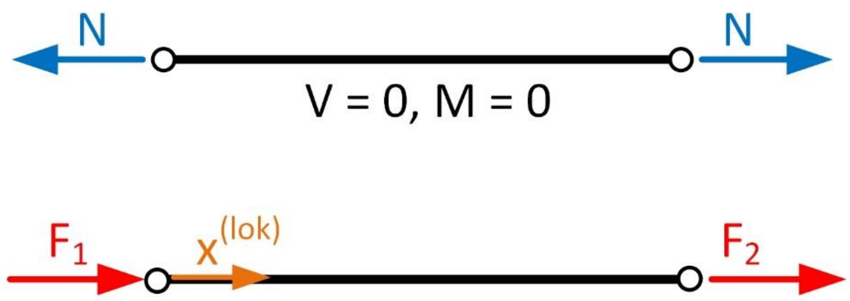
Properties of the element stiffness matrix:

- symmetric
- singular, i.e. the „structural system“ is kinematic

Element stiffness matrix of a truss element

Element section force matrix of a truss element in local coordinates

The element section forces are computed with the element section force matrix (or the element stress matrix for the stresses) after the nodal displacements for the global system have been determined.



Element forces

$$N = F_2 \qquad \text{with}$$

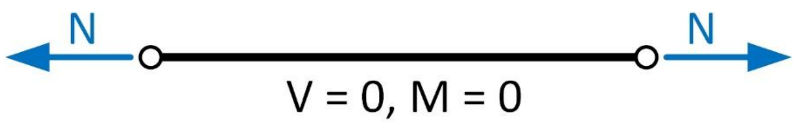
$$F_2 = \frac{E \cdot A}{\ell}\left(-u_1 + u_2\right)$$

Element section force matrix

$$N = \frac{E \cdot A}{\ell}\begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Element stiffness matrix of a truss element

Section forces matrix of a truss element in local coordinates



Normal force

$$N = \frac{E \cdot A}{\ell} \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$N = \underline{S}^{(\mathrm{lok})} \cdot \underline{u}^{(\mathrm{lok})}$$

$\underline{S}^{(\mathrm{lok})}$ **Section force matrix**

Normal stress

$$\sigma = \frac{E}{\ell} \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

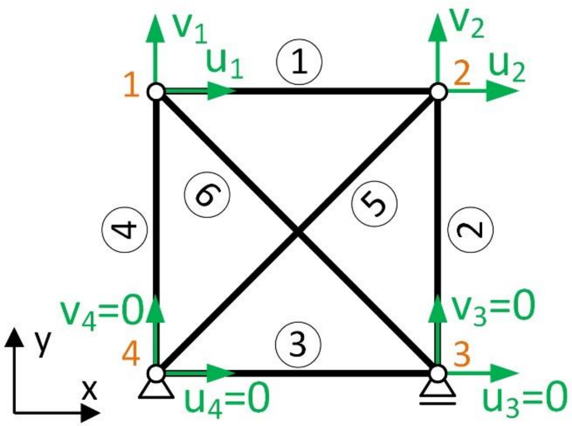
$$\sigma = \underline{S}_\sigma^{(\mathrm{lok})} \cdot \underline{u}^{(\mathrm{lok})}$$

$\underline{S}_\sigma^{(\mathrm{lok})}$ **Element stress matrix**

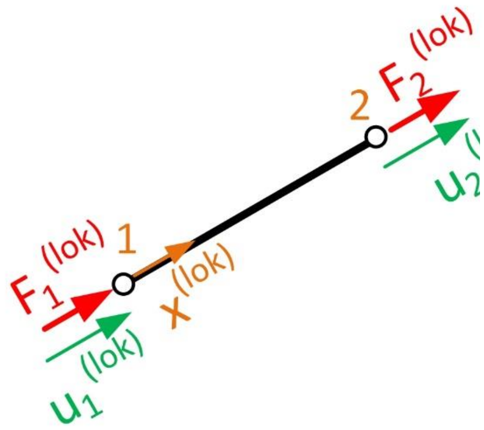
Element stiffness matrix of a truss element

Coordinate transformation

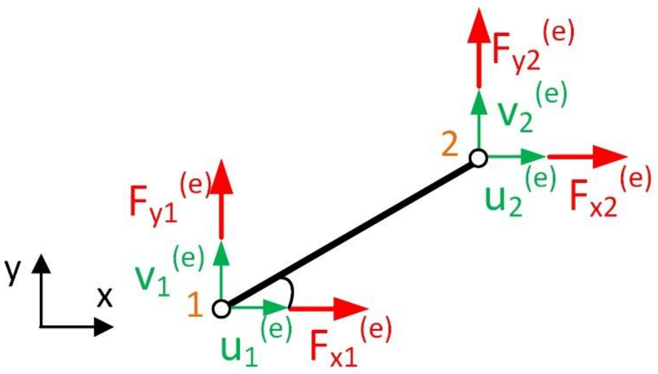
Element forces and displacements in global and local coordinates



Truss system



Element in local coodinates



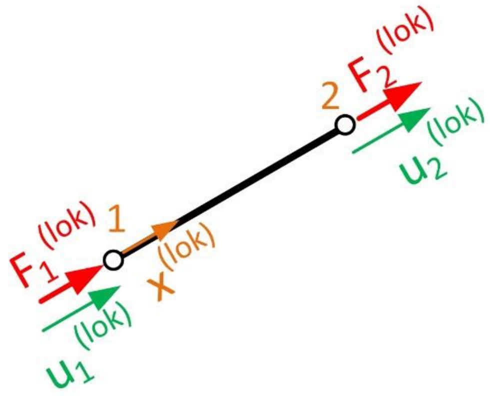
Element in global coordinates

Element stiffness matrix of a truss element

Coordinate transformation

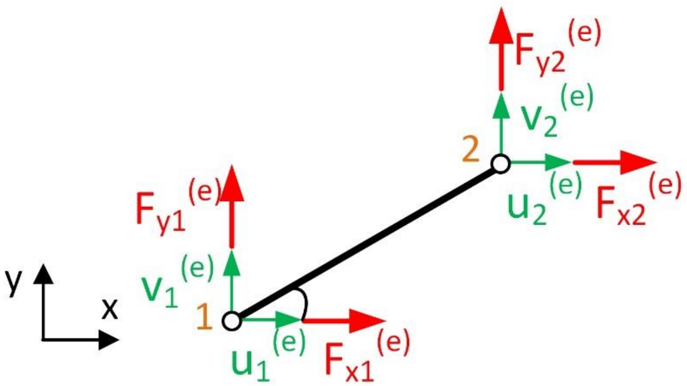
Element forces and displacements in local coordinates

$$\underline{u}^{(lok)} = \begin{bmatrix} u_1^{(\mathrm{lok})} \\ u_2^{(\mathrm{lok})} \end{bmatrix} \qquad \underline{F}^{(\mathrm{lok})} = \begin{bmatrix} F_1^{(\mathrm{lok})} \\ F_2^{(\mathrm{lok})} \end{bmatrix}$$



Element forces and displacements in global coordi

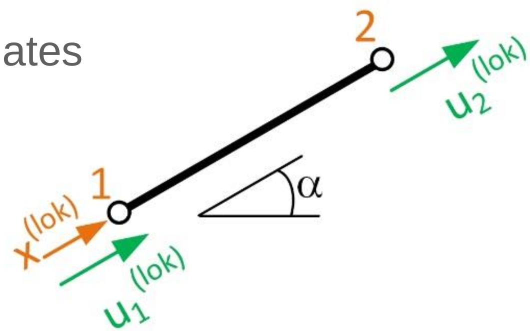
$$\underline{u}^{(e)} = \begin{bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \end{bmatrix} \qquad \underline{F}^{(e)} = \begin{bmatrix} F_{x1}^{(e)} \\ F_{y1}^{(e)} \\ F_{x2}^{(e)} \\ F_{y2}^{(e)} \end{bmatrix}$$

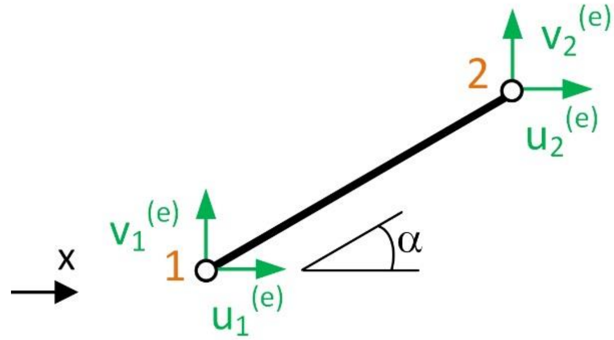


Element stiffness matrix of a truss element

Coordinate transformation: nodal point displacements

Local coordinates (element)

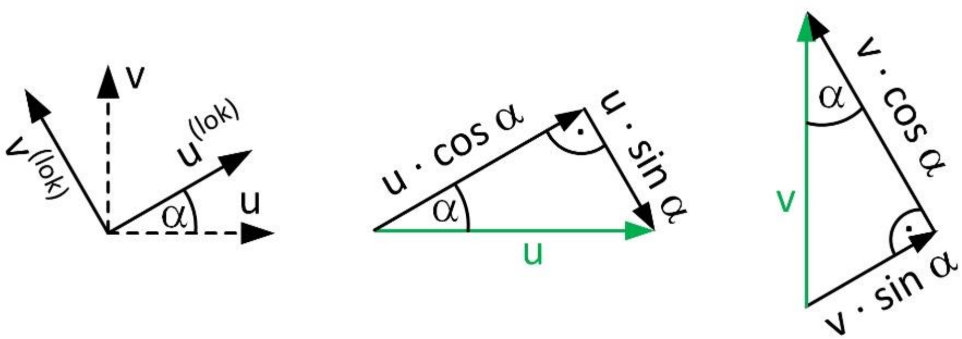




Global coordinates (system)

coordinate transformation

coordinate transformation - truss element



$$u^{(\mathrm{lok})} = u \cdot \cos\alpha + v \cdot \sin\alpha$$
$$v^{(\mathrm{lok})} = -u \cdot \sin\alpha + v \cdot \cos\alpha$$

$$u_1^{(\mathrm{lok})} = u_1^{(e)} \cdot \cos\alpha + v_1^{(e)} \cdot \sin\alpha$$
$$u_2^{(\mathrm{lok})} = u_2^{(e)} \cdot \cos\alpha + v_2^{(e)} \cdot \sin\alpha$$

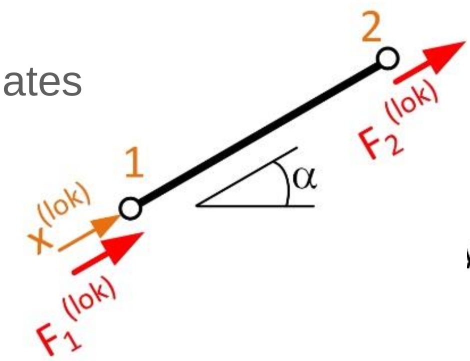
$$\begin{bmatrix} u_1^{(\mathrm{lok})} \\ u_2^{(\mathrm{lok})} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix} \cdot \begin{bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \end{bmatrix}$$

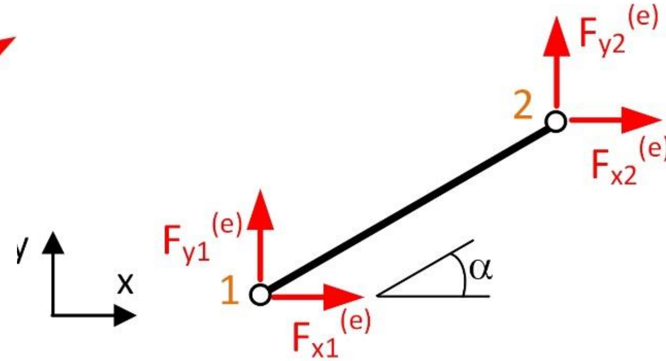
$$\underline{u}^{(\mathrm{lok})} = \underline{T} \cdot \underline{u}^{(e)}$$

Element stiffness matrix of a truss element

Coordinate transformation: nodal point forces

Local coordinates (element)

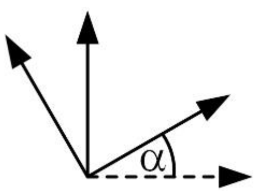


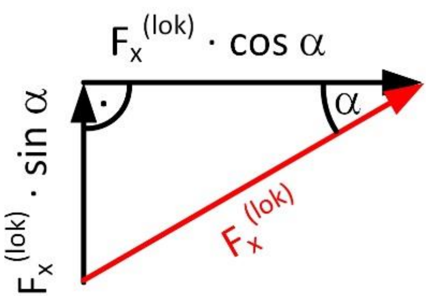


Global coordinates (system)

coordinate transformation

coordinate transformation - truss element





$$F_x = F_x^{(lok)} \cdot \cos\alpha$$

$$F_y = F_x^{(lok)} \cdot \sin\alpha$$

$$F_{x1}^{(e)} = \cos\alpha \cdot F_1^{(lok)}$$

$$F_{y1}^{(e)} = \sin\alpha \cdot F_1^{(lok)}$$

$$F_{x2}^{(e)} = \cos\alpha \cdot F_2^{(lok)}$$

$$F_{y2}^{(e)} = \sin\alpha \cdot F_2^{(lok)}$$

$$\Rightarrow \begin{bmatrix} F_{x1}^{(e)} \\ F_{y1}^{(e)} \\ F_{x2}^{(e)} \\ F_{y2}^{(e)} \end{bmatrix} = \begin{bmatrix} \cos\alpha & o \\ \sin\alpha & o \\ o & \cos\alpha \\ o & \sin\alpha \end{bmatrix} \cdot \begin{bmatrix} F_1^{(lok)} \\ F_2^{(lok)} \end{bmatrix}$$

$$\underline{F}^{(e)} = \underline{T}^{T} \cdot \underline{F}^{(lok)}$$

Element stiffness matrix of a truss element

Coordinate transformation: element stiffness matrix

$$\begin{bmatrix} F_1^{(lok)} \\ F_2^{(lok)} \end{bmatrix} = \frac{E \cdot A}{\ell} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1^{(lok)} \\ u_2^{(lok)} \end{bmatrix}$$

Coordinate transformation of nodal forces

substitution

$$\underline{F}^{(lok)} = \underline{K}^{(lok)} \cdot \underline{u}^{(lok)}$$

substitution

Coordinatetransformation of nodal point displacements

$$\begin{bmatrix} F_{x1}^{(e)} \\ F_{y1}^{(e)} \\ F_{x2}^{(e)} \\ F_{y2}^{(e)} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \\ 0 & \cos\alpha \\ 0 & \sin\alpha \end{bmatrix} \cdot \begin{bmatrix} F_1^{(lok)} \\ F_2^{(lok)} \end{bmatrix}$$

$$\begin{bmatrix} u_1^{(lok)} \\ u_2^{(lok)} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix} \cdot \begin{bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \end{bmatrix}$$

$$\underline{F}^{(e)} = \underline{T}^T \cdot \underline{F}^{(lok)}$$

$$\underline{F}^{(e)} = \underline{T}^T \cdot \underline{K}^{(lok)} \cdot \underline{T} \cdot \underline{u}^{(e)} \quad \text{or} \quad \underline{F}^{(e)} = \underline{K}^{(e)} \cdot \underline{u}^{(e)} \quad \text{with} \quad \underline{K}^{(e)} = \underline{T}^T \cdot \underline{K}^{(lok)} \cdot \underline{T}$$

Element stiffness matrix of a truss element

Coordinate transformation: element stiffness matrix

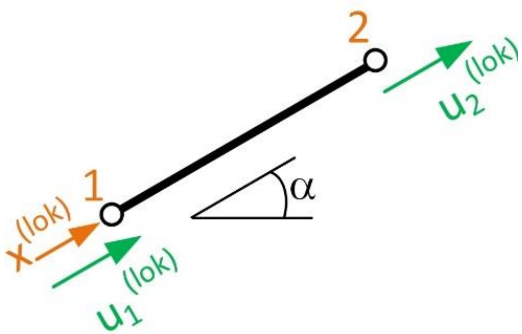
$$\begin{bmatrix}F_{x1}^{(e)}\\F_{y1}^{(e)}\\F_{x2}^{(e)}\\F_{y2}^{(e)}\end{bmatrix} = \frac{E\cdot A}{\ell}\underbrace{\begin{bmatrix}\cos\alpha & 0\\ \sin\alpha & 0\\ 0 & \cos\alpha\\ 0 & \sin\alpha\end{bmatrix}}_{T^{T}}\cdot\underbrace{\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}_{K^{(lok)}}\cdot\underbrace{\begin{bmatrix}\cos\alpha & \sin\alpha & 0 & 0\\ 0 & 0 & \cos\alpha & \sin\alpha\end{bmatrix}}_{T}\cdot\begin{bmatrix}u_1^{(e)}\\v_1^{(e)}\\u_2^{(e)}\\v_2^{(e)}\end{bmatrix}$$

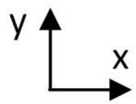
$$\underline{K}^{(e)} = \frac{E\cdot A}{\ell}\cdot\begin{bmatrix}\cos^2\alpha & \sin\alpha\cdot\cos\alpha & -\cos^2\alpha & -\sin\alpha\cdot\cos\alpha\\ \sin\alpha\cdot\cos\alpha & \sin^2\alpha & -\sin\alpha\cdot\cos\alpha & -\sin^2\alpha\\ -\cos^2\alpha & -\sin\alpha\cdot\cos\alpha & \cos^2\alpha & \sin\alpha\cdot\cos\alpha\\ -\sin\alpha\cdot\cos\alpha & -\sin^2\alpha & \sin\alpha\cdot\cos\alpha & \sin^2\alpha\end{bmatrix}$$

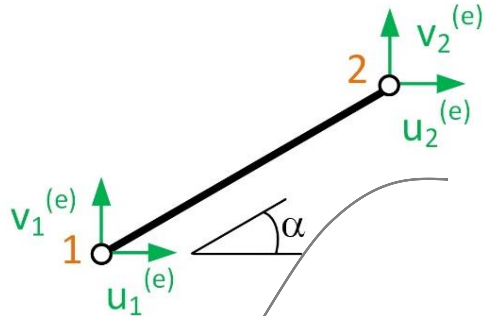
$$\underline{F}^{(e)} = \underline{T}^{T}\cdot\underline{K}^{(lok)}\cdot\underline{T}\cdot\underline{u}^{(e)} \quad \text{or} \quad \underline{F}^{(e)} = \underline{K}^{(e)}\cdot\underline{u}^{(e)} \quad \text{with} \quad \underline{K}^{(e)} = \underline{T}^{T}\cdot\underline{K}^{(lok)}\,\underline{T}$$

Element stiffness matrix of a truss element

Coordinate transformation: element section force matrix







$$\begin{bmatrix} u_1^{(\text{lok})} \\ u_2^{(\text{lok})} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & o & o \\ o & o & \cos\alpha & \sin\alpha \end{bmatrix} \cdot \begin{bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \end{bmatrix}$$

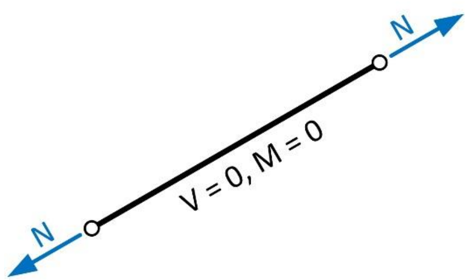
Coordinate transformation of nodal displacements

Section forces in local coordinates

$$N = \frac{E \cdot A}{\ell} \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1^{(\text{lok})} \\ u_2^{(\text{lok})} \end{bmatrix}$$

Section forces matrix in global coordinates

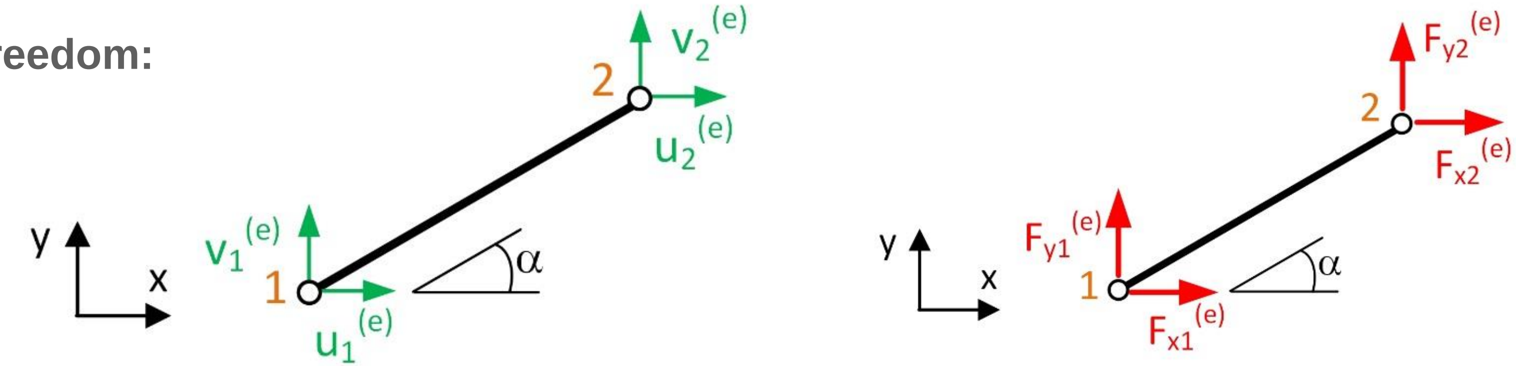
$$N = \frac{E \cdot A}{\ell} \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & \sin\alpha & o & o \\ o & o & \cos\alpha & \sin\alpha \end{bmatrix} \cdot \begin{bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \end{bmatrix}$$



$$N = \underline{S}^{(e)} \cdot \underline{u}^{(e)}$$

Element stiffness matrix of a truss element

Degrees of freedom:



Stiffness matrix:

$$\frac{E \cdot A}{\ell} \cdot \begin{bmatrix} \cos^2\alpha & \sin\alpha\cdot\cos\alpha & -\cos^2\alpha & -\sin\alpha\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha & \sin^2\alpha & -\sin\alpha\cdot\cos\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\sin\alpha\cdot\cos\alpha & \cos^2\alpha & \sin\alpha\cdot\cos\alpha \\ -\sin\alpha\cdot\cos\alpha & -\sin^2\alpha & \sin\alpha\cdot\cos\alpha & \sin^2\alpha \end{bmatrix} \cdot \begin{bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \end{bmatrix} = \begin{bmatrix} F_{x1}^{(e)} \\ F_{y1}^{(e)} \\ F_{x2}^{(e)} \\ F_{y2}^{(e)} \end{bmatrix}$$

$$\underline{F}^{(e)} = \underline{K}^{(e)} \cdot \underline{u}^{(e)} \quad with \quad \underline{K}^{(e)} = \underline{T}^{T} \cdot \underline{K}^{(lok)} \underline{T}$$

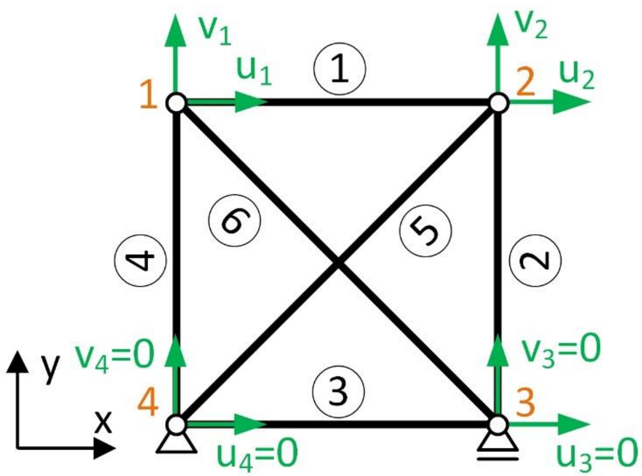
Section force matrix for normal force:

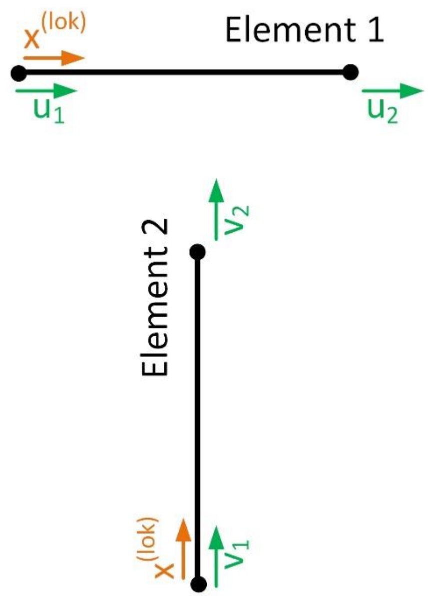
$$N = \underline{S}^{(e)} \cdot \underline{u}^{(e)}$$

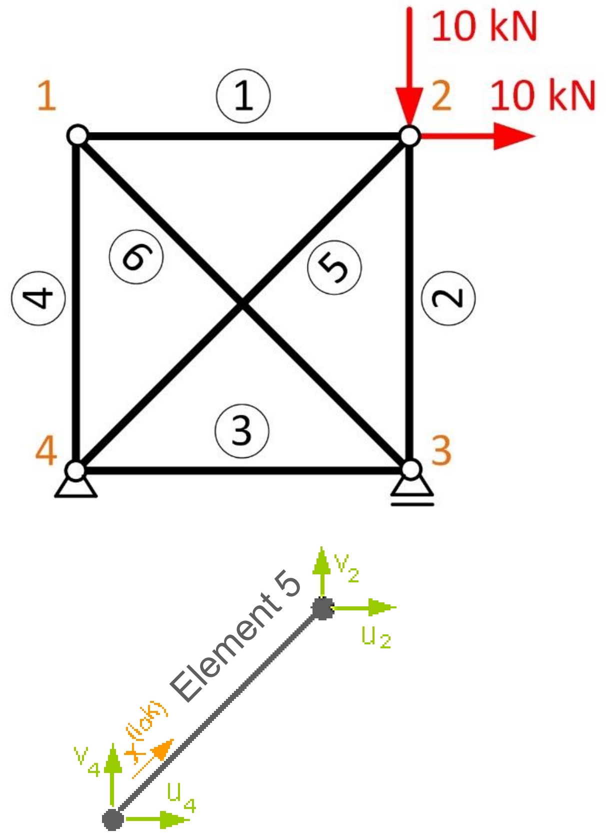
$$N = \frac{E \cdot A}{\ell} \cdot \begin{bmatrix} -\cos\alpha & -\sin\alpha & \cos\alpha & \sin\alpha \end{bmatrix} \cdot \begin{bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \end{bmatrix}$$

Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices







$E = 2.1 \cdot 10^8 \text{ kN/m}^2 \quad A = 0.004 \text{ m}^2$

$\ell_1 - \ell_4 = 3.00 \text{m} \qquad \ell_5 - \ell_6 = 3.00 \cdot \sqrt{2} = 4.24 \text{ m}$

Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices

Element 1: $\alpha = 0°$

$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \cdot \frac{0.004}{3.00} = 2.8 \cdot 10^5$$

Element stiffness matrix

$$\begin{bmatrix} F_{x1}^{(1)} \\ F_{x2}^{(1)} \end{bmatrix} = 2.80 \cdot 10^5 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Element section force matrices

$$N = 2.8 \cdot 10^5 \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

System

Stiffness matrix

Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices

Element 2: $\alpha = 90°$

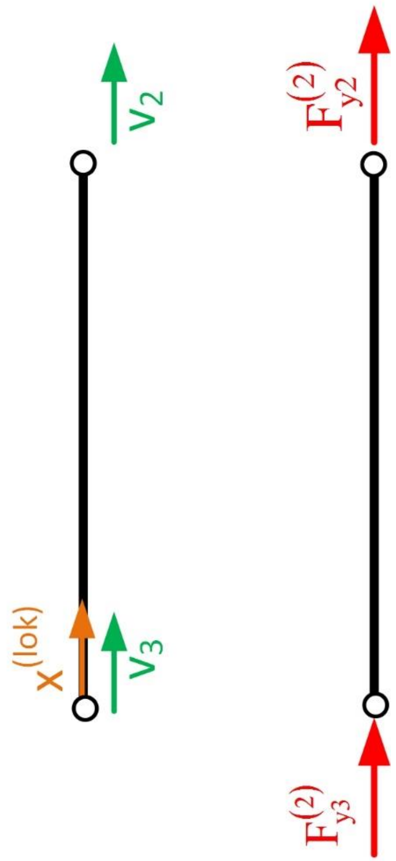
$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \cdot \frac{0.004}{3.00} = 2.8 \cdot 10^5$$

Element stiffness matrix

$$\begin{bmatrix} F_{y3}^{(2)} \\ F_{y2}^{(2)} \end{bmatrix} = 2.80 \cdot 10^5 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_3 \\ v_2 \end{bmatrix}$$

Element section force matrices

$$N = 2.8 \cdot 10^5 \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_3 \\ v_2 \end{bmatrix}$$



System

Stiffness matrix

Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices

Element 3: $\alpha = 0°$

$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \cdot \frac{0.004}{3.00} = 2.8 \cdot 10^5$$

Element stiffness matrix

$$\begin{bmatrix} F_{x4}^{(3)} \\ F_{x3}^{(3)} \end{bmatrix} = 2.80 \cdot 10^5 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_4 \\ u_3 \end{bmatrix}$$

Element section force matrices

$$N = \quad 2.8 \cdot 10^5 \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_4 \\ u_3 \end{bmatrix}$$

System

Stiffness matrix

Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices

Element 4: $\alpha = 90°$

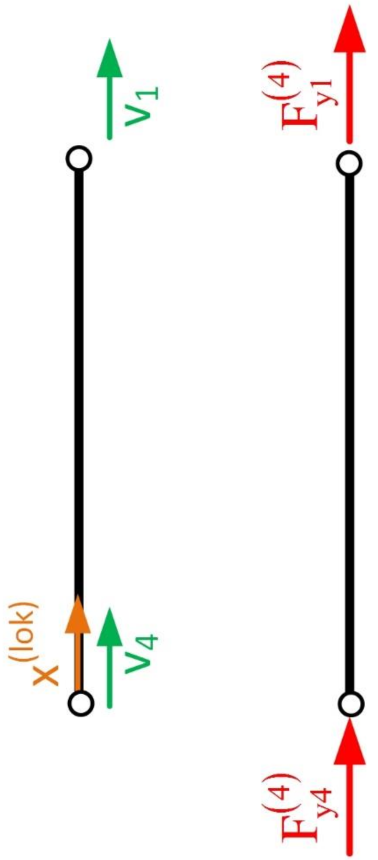
$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \cdot \frac{0.004}{3.00} = 2.8 \cdot 10^5$$

Element stiffness matrix

$$\begin{bmatrix} F_{y4}^{(4)} \\ F_{y1}^{(4)} \end{bmatrix} = 2.80 \cdot 10^5 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_4 \\ v_1 \end{bmatrix}$$

Element section force matrices

$$N = 2.8 \cdot 10^5 \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_4 \\ v_1 \end{bmatrix}$$

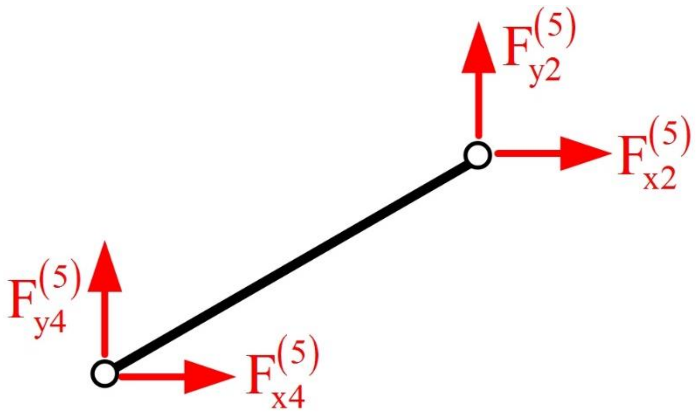


System

Stiffness matrix

Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices

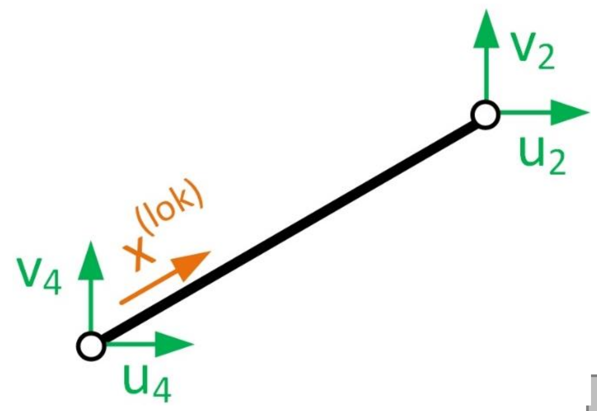


Element 5: $\alpha = 45°$

$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \, \frac{0.004}{4.24} = 1.98 \cdot 10^5$$

Element stiffness matrix

$$\begin{bmatrix} F_{x4}^{(5)} \\ F_{y4}^{(5)} \\ F_{x2}^{(5)} \\ F_{y2}^{(5)} \end{bmatrix} = 1.98 10^5 \cdot \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} u_4 \\ v_4 \\ u_2 \\ v_2 \end{bmatrix}$$



Element section force matrices

$$N = 1.98 \cdot 10^5 \cdot [-0.707 \;\; -0.707 \;\; 0.707 \;\; 0.707] \cdot \begin{bmatrix} u_4 \\ v_4 \\ u_2 \\ v_2 \end{bmatrix}$$

System

Stiffness matrix

Element stiffness matrix of a truss element

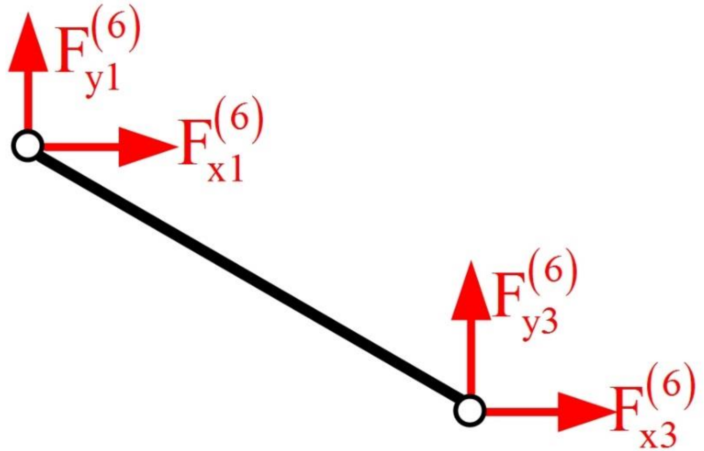
Example: Element stiffness and section force matrices

Element 6: $\alpha = 135°$

$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \, \frac{0.004}{4.24} = 1.98 \cdot 10^5$$

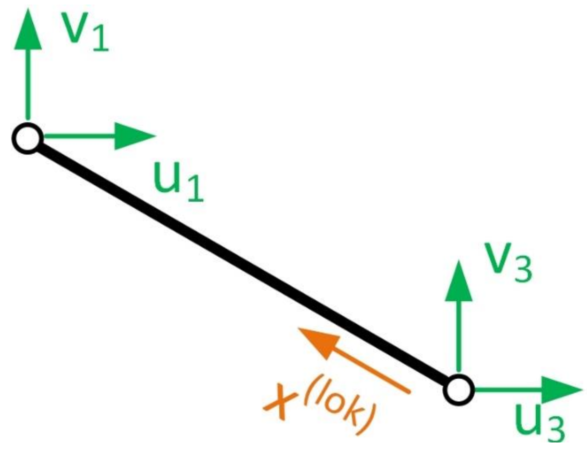
Element stiffness matrix

$$\begin{bmatrix} F_{x3}^{(6)} \\ F_{y3}^{(6)} \\ F_{x1}^{(6)} \\ F_{y1}^{(6)} \end{bmatrix} = 1.98 \cdot 10^5 \cdot \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} u_3 \\ v_3 \\ u_1 \\ v_1 \end{bmatrix}$$



Element section force matrices

$$N = 1.98 \cdot 10^5 \cdot \begin{bmatrix} 0.707 & -0.707 & -0.707 & 0.707 \end{bmatrix} \cdot \begin{bmatrix} u_3 \\ v_3 \\ u_1 \\ v_1 \end{bmatrix}$$



System

Stiffness matrix

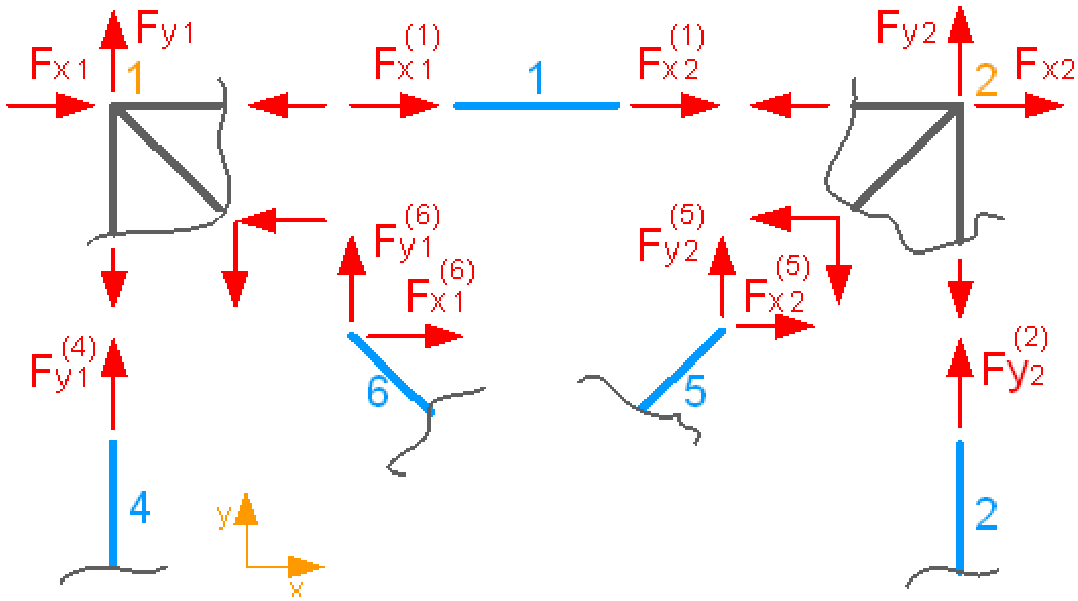
Global stiffness matrix

Assembly of the global stiffness matrix

The global stiffness matrix is constructed by assembling the elements at the nodal points.

Compatibility conditions

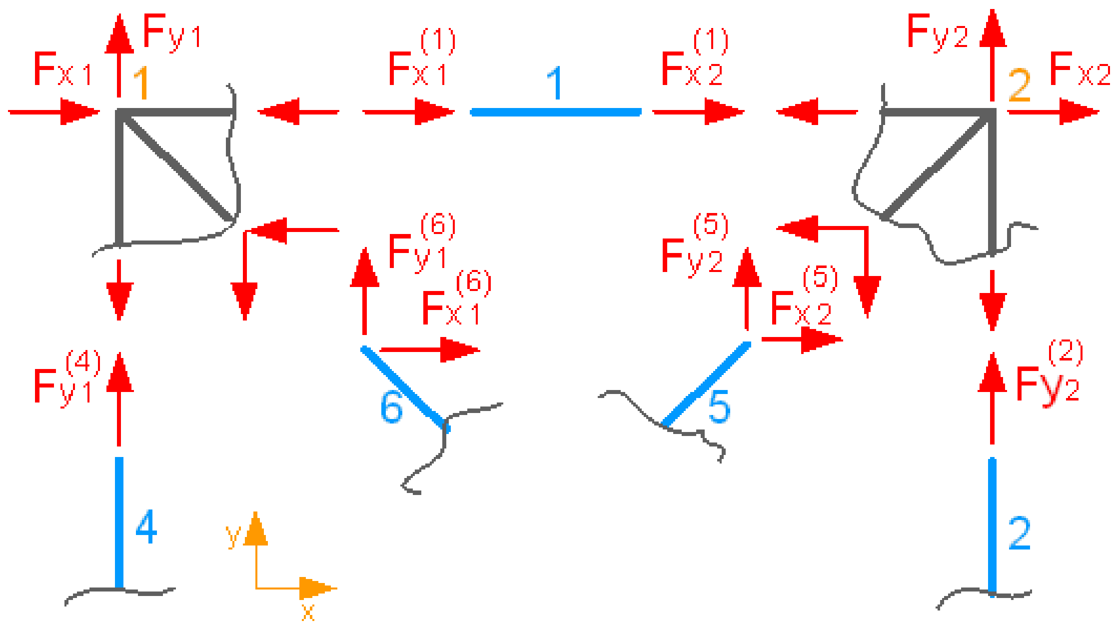
- Equations of equilibrium at all nodal points
- Compatibility of the displacements at all nodal points



Global stiffness matrix

Assembly of the global stiffness matrix

Introductory Example



Equilibrium at nodal point 1:

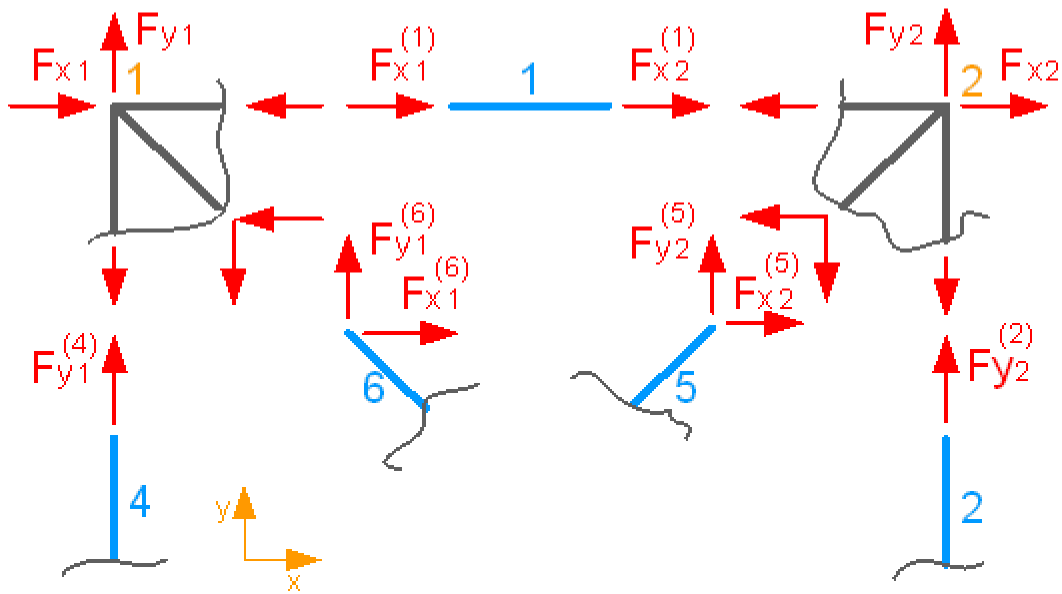
$$\sum X_i = 0 \Rightarrow F_{x1}^{(1)} + F_{x1}^{(6)} = F_{x1} \quad (1a)$$

$$\sum Y_i = 0 \Rightarrow F_{y1}^{(4)} + F_{y1}^{(6)} = F_{y1} \quad (1b)$$

Global stiffness matrix

Assembly of the global stiffness matrix

Introductory Example



Equilibrium at nodal point 2:

$$\sum X_i = 0 \Rightarrow F_{x2}^{(1)} + F_{x2}^{(5)} = F_{x2} \quad (2a)$$

$$\sum Y_i = 0 \Rightarrow F_{y2}^{(2)} + F_{y2}^{(5)} = F_{y2} \quad (2b)$$

Global stiffness matrix

Assembly of the global stiffness matrix

Nodal point 1 → $\sum_i F_{x1}^{(i)} = F_{x1}$

$\sum_i F_{y1}^{(i)} = F_{y1}$

$\bullet$

$\bullet$

$\bullet$

Nodal point n → $\sum_n F_{yn}^{(i)} = F_{yn}$

$$\Rightarrow \sum \begin{bmatrix} F_{x1}^{(i)} \\ F_{y1}^{(i)} \\ \bullet \\ \bullet \\ \bullet \\ F_{yn}^{(i)} \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ \bullet \\ \bullet \\ \bullet \\ F_{yn} \end{bmatrix}$$

$$\sum_i \underline{F}^{(i)} = \underline{F}$$

Element forces of the truss elements

External forces (loads)

Element forces

External forces (loads)

Global stiffness matrix

Assembly of the global stiffness matrix

The element forces are expressed by the element stiffness matrices.

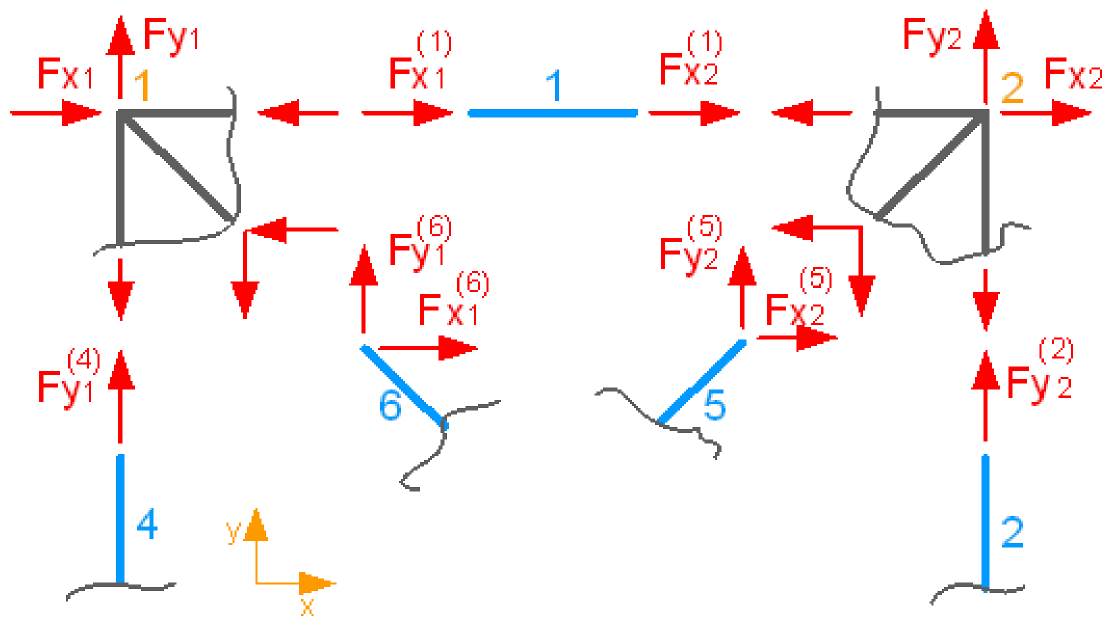
Introductory example

element forces at nodal point 1

- Element 1
- Element 4
- Element 6

element forces at nodal point 2

- Element 1
- Element 2
- Element 5



Global stiffness matrix

Assembly of the global stiffness matrix

The element stiffness matrices are expanded with zeroes for all degrees of freedom of the system.

Introductory example, element 1: $2.80 \cdot 10^5 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_{x1}^{(1)} \\ F_{x2}^{(1)} \end{bmatrix}$ $\quad \underline{K}^{(1)} \cdot \underline{u} = \underline{F}^{(1)}$

$$2.8 \cdot 10^5 \cdot \begin{array}{c} \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 1.0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \cdot \begin{array}{c} u \\ \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} \end{array} = \begin{bmatrix} F_{x1}^{(1)} \\ F_{y1}^{(1)} \\ F_{x2}^{(1)} \\ F_{y2}^{(1)} \\ F_{x3}^{(1)} \\ F_{y3}^{(1)} \\ F_{x4}^{(1)} \\ F_{y4}^{(1)} \end{bmatrix}$$

$$\hat{\underline{K}}^{(1)} \cdot \underline{u} = \hat{\underline{F}}^{(1)}$$

Element 1

Global stiffness matrix

Assembly of the global stiffness matrix

The element stiffness matrices are expanded with zeroes for all degrees of freedom of the system.

Introductory example, element 2:
$$\begin{bmatrix} F_{y3}^{(2)} \\ F_{y2}^{(2)} \end{bmatrix} = 2.80 \cdot 10^5 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_3 \\ v_2 \end{bmatrix} \qquad \underline{K}^{(2)} \cdot \underline{u} = \underline{F}^{(2)}$$

$$\begin{array}{cc} & \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ 2.8 \cdot 10^5 \cdot & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & -1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} F_{x1}^{(2)} \\ F_{y1}^{(2)} \\ F_{x2}^{(2)} \\ F_{y2}^{(2)} \\ F_{x3}^{(2)} \\ F_{y3}^{(2)} \\ F_{x4}^{(2)} \\ F_{y4}^{(2)} \end{bmatrix} \qquad \underline{\hat{K}}^{(2)} \cdot \underline{u} = \underline{\hat{F}}^{(2)}$$

Element 2

Global stiffness matrix

Assembly of the global stiffness matrix

Expanded matrix for element **i** : $\underline{\hat{F}}^{(i)} = \underline{\hat{K}}^{(i)} \cdot \underline{u}$

Sum over all elements: $\sum_i \underline{\hat{F}}^{(i)} = \underline{F}$ $\qquad \sum \underline{\hat{K}}^{(i)} \cdot \underline{u} = \underline{F}$

Element forces

External forces (loads)

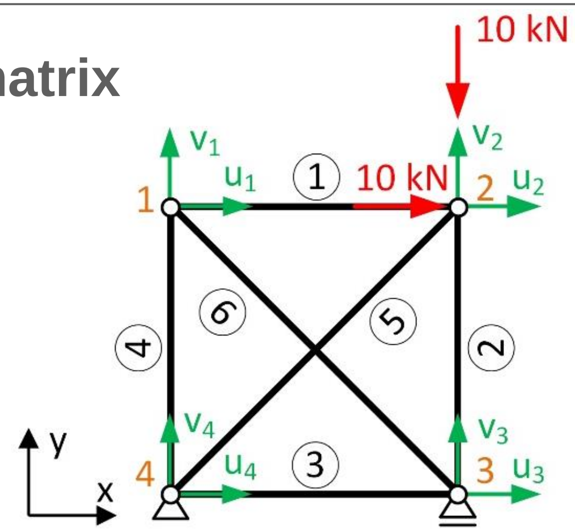
Global stiffness matrix $\underline{K} \cdot \underline{u} = \underline{F} \quad with \quad \underline{K} = \sum \underline{\hat{K}}^{(i)}$

The global stiffness matrix is assembled from the element stiffness matrix.

The coefficients of the element stiffness matrix are added to the global stiffness matrix at the rows and columns corresponding to their degress of freedom.

Global stiffness matrix

Assembly of the global stiffness matrix



Element 1

$$\begin{bmatrix} F_{x1}^{(1)} \\ F_{x2}^{(1)} \end{bmatrix} = 2.80\cdot 10^5 \cdot \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$2.8\cdot 10^5 \cdot \begin{array}{c} \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 1.0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}$$

Element 1

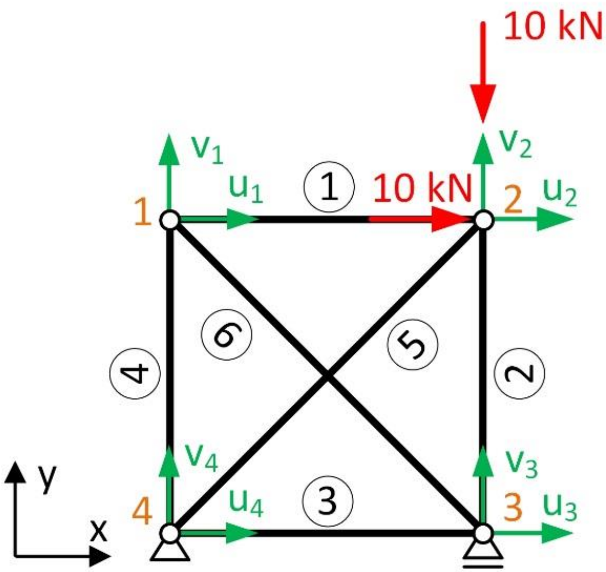
Global stiffness matrix

Assembly of the global stiffness matrix

Element 2

$$\begin{bmatrix} F_{y3}^{(2)} \\ F_{y2}^{(2)} \end{bmatrix} = 2.80\cdot 10^5 \cdot \begin{array}{cc} v_3 & v_2 \\ \left[\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right] \end{array} \cdot \begin{bmatrix} v_3 \\ v_2 \end{bmatrix}$$

$$2.8\cdot 10^5 \cdot \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \\ \left[\begin{matrix} 1.0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & -1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right] \end{array} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}$$



Element 2

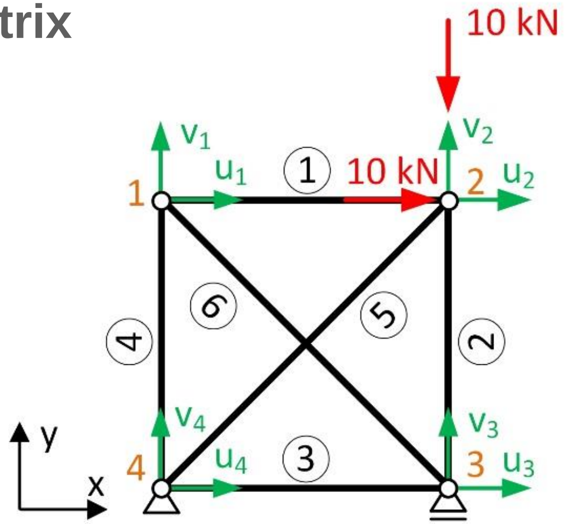
Global stiffness matrix

Assembly of the global stiffness matrix

Element 3

$$\begin{bmatrix} F_{x4}^{(3)} \\ F_{x3}^{(3)} \end{bmatrix} = 2.80 \cdot 10^5 \cdot \begin{array}{c} \begin{array}{cc} u_4 & u_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \cdot \begin{bmatrix} u_4 \\ u_3 \end{bmatrix}$$

$$2.8 \cdot 10^5 \cdot \begin{array}{c} \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 1.0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & -1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 & -1.0 & 0 \\ 0 & 0 & 0 & -1.0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}$$



Element 3

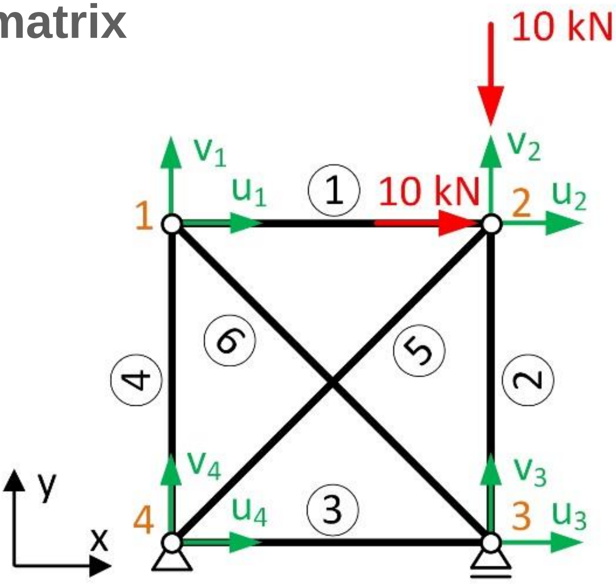
Global stiffness matrix

Assembly of the global stiffness matrix

Element 4

$$\begin{bmatrix} F_{y4}^{(4)} \\ F_{y1}^{(4)} \end{bmatrix} = 2.80\cdot 10^5 \cdot \begin{array}{cc} v_4 & v_1 \\ \left[\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right] \end{array} \cdot \begin{bmatrix} v_4 \\ v_1 \end{bmatrix}$$

$$2.8\cdot 10^5 \cdot \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \\ \left[\begin{matrix} 1.0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1.0} & 0 & 0 & 0 & 0 & 0 & \mathbf{-1.0} \\ -1.0 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & -1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 & -1.0 & 0 \\ 0 & 0 & 0 & -1.0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.0 & 0 & 1.0 & 0 \\ 0 & \mathbf{-1.0} & 0 & 0 & 0 & 0 & 0 & \mathbf{1.0} \end{matrix}\right] \end{array} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}$$



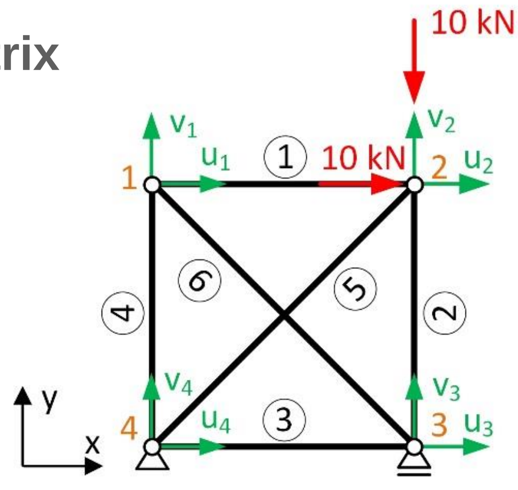
Element 4

Global stiffness matrix

Assembly of the global stiffness matrix

Element 5:

$$
\begin{bmatrix} F_{x4}^{(5)} \\ F_{y4}^{(5)} \\ F_{x2}^{(5)} \\ F_{y2}^{(5)} \end{bmatrix}
= 2.8 \cdot 10^5 \cdot
\begin{array}{c}
\begin{array}{cccc} u_4 & v_4 & u_2 & v_2 \end{array} \\
\begin{bmatrix}
0.35 & 0.35 & -0.35 & -0.35 \\
0.35 & 0.35 & -0.35 & -0.35 \\
-0.35 & -0.35 & 0.35 & 0.35 \\
-0.35 & -0.35 & 0.35 & 0.35
\end{bmatrix}
\end{array}
\cdot
\begin{bmatrix} u_4 \\ v_4 \\ u_2 \\ v_2 \end{bmatrix}
$$



$$
2.8 \cdot 10^5 \cdot
\begin{array}{c}
\begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\
\begin{bmatrix}
1.0 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0 & 0 & 0 & 0 & 0 & 0 & -1.0 \\
-1.0 & 0 & 1+0.35 & 0.35 & 0 & 0 & -0.35 & -0.35 \\
0 & 0 & 0.35 & 1+0.35 & 0 & -1.0 & -0.35 & -0.35 \\
0 & 0 & 0 & 0 & 1.0 & 0 & -1.0 & 0 \\
0 & 0 & 0 & -1.0 & 0 & 1.0 & 0 & 0 \\
0 & 0 & -0.35 & -0.35 & -1.0 & 0 & 1+0.35 & 0.35 \\
0 & -1.0 & -0.35 & -0.35 & 0 & 0 & 0.35 & 1+0.35
\end{bmatrix}
\end{array}
\cdot
\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}
=
\begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}
$$

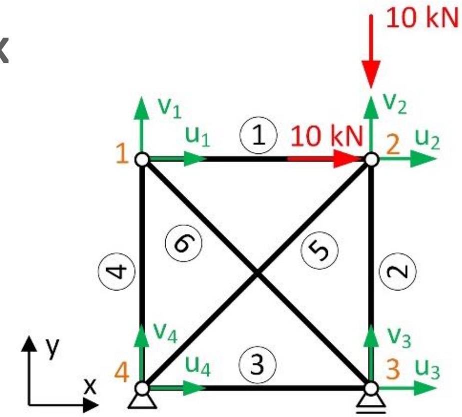
Element 5

Global stiffness matrix

Assembly of the global stiffness matrix

Element 6:

$$
\begin{bmatrix} F_{x3}^{(6)} \\ F_{y3}^{(6)} \\ F_{x1}^{(6)} \\ F_{y1}^{(6)} \end{bmatrix} = 2.8\cdot 10^5 \cdot
\begin{array}{c}
\begin{array}{cccc} u_3 & v_3 & u_1 & v_1 \end{array} \\
\begin{bmatrix} 0.35 & -0.35 & -0.35 & 0.35 \\ -0.35 & 0.35 & 0.35 & -0.35 \\ -0.35 & 0.35 & 0.35 & -0.35 \\ 0.35 & -0.35 & -0.35 & 0.35 \end{bmatrix}
\end{array}
\cdot \begin{bmatrix} u_3 \\ v_3 \\ u_1 \\ v_1 \end{bmatrix}
$$



Element 6

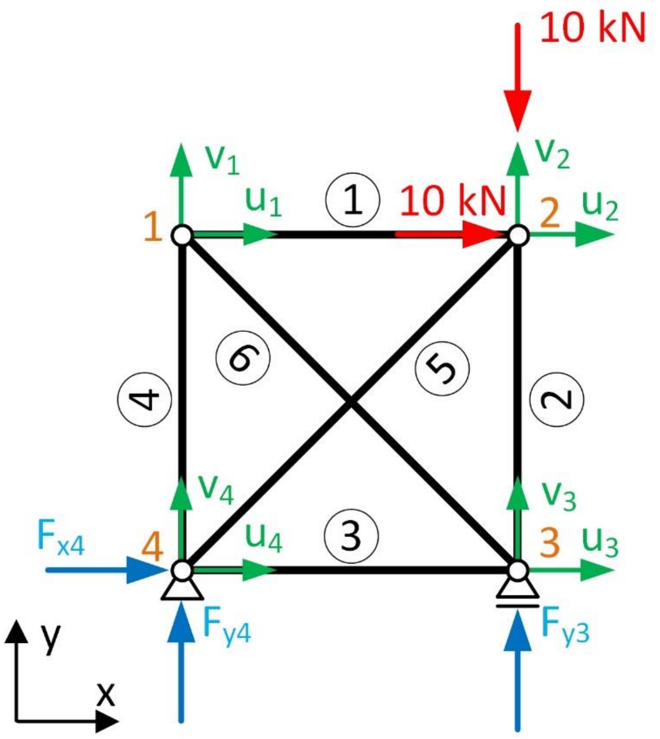
$$
2.8\cdot 10^5 \cdot
\begin{array}{c}
\begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\
\begin{bmatrix}
1+0.35 & -0.35 & -1.0 & 0 & -0.35 & 0.35 & 0 & 0 \\
-0.35 & 1+0.35 & 0 & 0 & 0.35 & -0.35 & 0 & -1.0 \\
-1.0 & 0 & 1.35 & 0.35 & 0 & 0 & -0.35 & -0.35 \\
0 & 0 & 0.35 & 1.35 & 0 & -1.0 & -0.35 & -0.35 \\
-0.35 & 0.35 & 0 & 0 & 1+0.35 & -0.35 & -1.0 & 0 \\
0.35 & -0.35 & 0 & -1.0 & -0.35 & 1+0.35 & 0 & 0 \\
0 & 0 & -0.35 & -0.35 & -1.0 & 0 & 1.35 & 0.35 \\
0 & -1.0 & -0.35 & -0.35 & 0 & 0 & 0.35 & 1.35
\end{bmatrix}
\end{array}
\cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}
$$

Global stiffness matrix

Assembly of the global stiffness matrix

Load vector

$$\begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10. \\ -10. \\ 0 \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}$$



Global stiffness matrix

Global stiffness matrix without restraints – introductory example

$$2.8\cdot 10^5 \cdot \begin{array}{c} \begin{matrix} \quad u_1 & \quad v_1 & \quad u_2 & \quad v_2 & \quad u_3 & \quad v_3 & \quad u_4 & \quad v_4 \end{matrix} \\ \begin{bmatrix} \boldsymbol{1.35} & \boldsymbol{-0.35} & \boldsymbol{-1.0} & 0 & \boldsymbol{-0.35} & 0.35 & 0 & 0 \\ \boldsymbol{-0.35} & \boldsymbol{1.35} & 0 & 0 & \boldsymbol{0.35} & -0.35 & 0 & -1.0 \\ \boldsymbol{-1.0} & 0 & \boldsymbol{1.35} & \boldsymbol{0.35} & 0 & 0 & -0.35 & -0.35 \\ 0 & 0 & \boldsymbol{0.35} & \boldsymbol{1.35} & 0 & -1.0 & -0.35 & -0.35 \\ \boldsymbol{-0.35} & \boldsymbol{0.35} & 0 & 0 & \boldsymbol{1.35} & -0.35 & -1.0 & 0 \\ 0.35 & -0.35 & 0 & -1.0 & -0.35 & 1.35 & 0 & 0 \\ 0 & 0 & -0.35 & -0.35 & -1.0 & 0 & 1.35 & 0.35 \\ 0 & -1.0 & -0.35 & -0.35 & 0 & 0 & 0.35 & 1.35 \end{bmatrix} \end{array} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \\ -10 \\ 0 \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}$$

Properties of the stiffness matrix

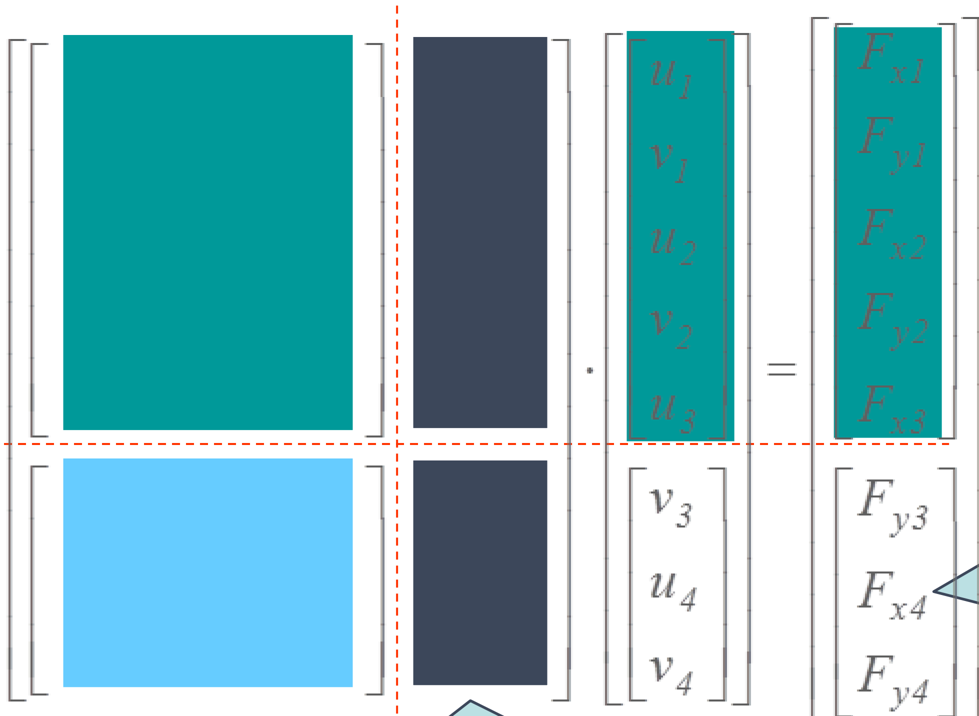
- symmetric (composed of symmetric element matrices)
- singular, because the structural system is (still) kinematic

System

Global stiffness matrix

Consideration of the support conditions

Introductory example

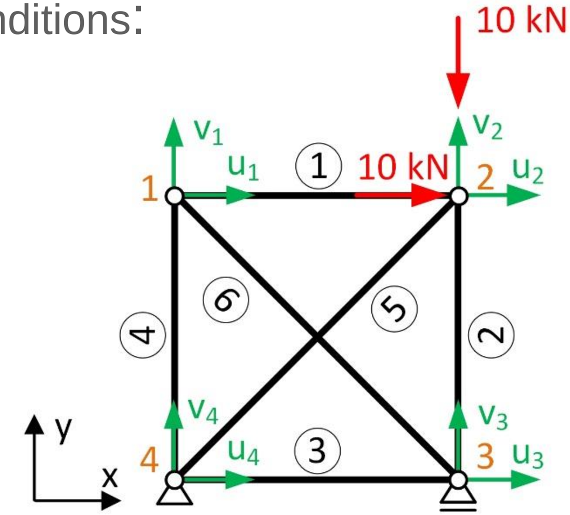


Support conditions:

$v_3 = 0$

$u_4 = 0$

$v_4 = 0$



Unknown restraint forces

⇒ rows have to be omitted in the system of equations

⇒ rows are later used to determine the support forces

Columns are multiplied with 0 and can be omitted

Global stiffness matrix

Stiffness matrix with consideration of the support conditions

System of equations for the displacements

$$2.8 \cdot 10^5 \cdot \begin{bmatrix} 1.35 & -0.35 & -1.0 & 0 & -0.35 \\ -0.35 & 1.35 & 0 & 0 & 0.35 \\ -1.0 & 0 & 1.35 & 0.35 & 0 \\ 0 & 0 & 0.35 & 1.35 & 0 \\ -0.35 & 0.35 & 0 & 0 & 1.35 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \\ -10 \\ 0 \end{bmatrix}$$

Properties of the global stiffness matrix:

- regular
- symmetric

Equations for the support forces

$$2.80 \cdot 10^5 \cdot \begin{bmatrix} 0.35 & -0.35 & 0 & -1.0 & -0.35 \\ 0 & 0 & -0.35 & -0.35 & -1.0 \\ 0 & -1.0 & -0.35 & -0.35 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}$$

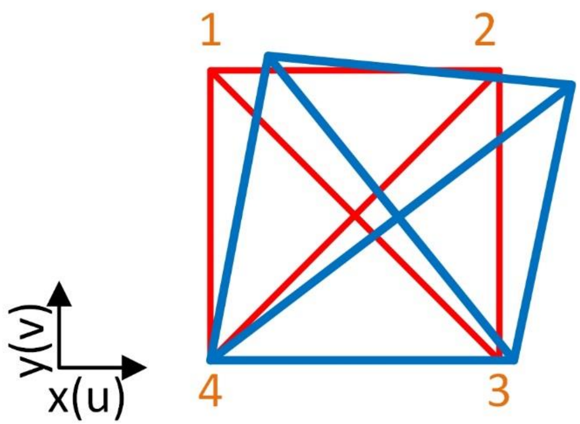
The support forces always fulfill the equilibrium conditions with the nodal forces.

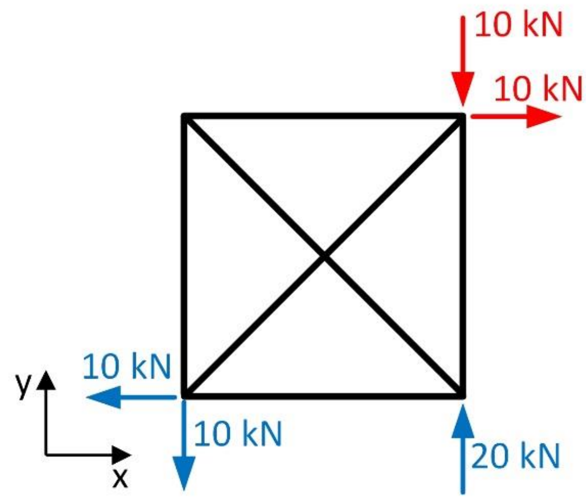
Introductory example

Results

Nodal point displacements:

$$\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.86 \\ 0.18 \\ 1.04 \\ -0.54 \\ 0.18 \end{bmatrix} \cdot 10^{-4}$$





Support forces:

$$2.80 \cdot 10^{5} \cdot \begin{bmatrix} 0.35 & -0.35 & 0 & -1. & -0.35 \\ 0 & 0 & -0.35 & -0.35 & -1. \\ 0 & -1. & -0.35 & -0.35 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.86 \\ 0.18 \\ 1.04 \\ -0.54 \\ 0.18 \end{bmatrix} \cdot 10^{-4} = \begin{bmatrix} 20.0 \\ -10.0 \\ -10.0 \end{bmatrix} = \begin{bmatrix} F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix} \quad [\mathrm{kN}]$$

Equilibrium control!

Introductory example

Results

Section forces:

Element 1:

Element 1

$$N_1 = 2.80 \cdot 10^5 \cdot \begin{bmatrix} -1. & 1. \end{bmatrix} \cdot \begin{bmatrix} 0.86 \\ 1.04 \end{bmatrix} \cdot 10^{-4} = 5.0$$

Element2:

Element 2

$$N_2 = 2.80 \cdot 10^5 \cdot \begin{bmatrix} -1. & 1. \end{bmatrix} \cdot \begin{bmatrix} 0.00 \\ -0.54 \end{bmatrix} \cdot 10^{-4} = -15.0$$

Element 3:

Element 3

$$N_3 = 2.80 \cdot 10^5 \cdot \begin{bmatrix} -1. & 1. \end{bmatrix} \cdot \begin{bmatrix} 0.00 \\ 0.18 \end{bmatrix} \cdot 10^{-4} = 5.0$$

Element 4:

Element 4

$$N_4 = 2.80 \cdot 10^5 \cdot \begin{bmatrix} -1. & 1. \end{bmatrix} \cdot \begin{bmatrix} 0.00 \\ 0.18 \end{bmatrix} \cdot 10^{-4} = 5.0$$

Element 5:

Element 5

$$N_5 = 1.98 \cdot 10^5 \cdot \begin{bmatrix} -0.71 & -0.71 & 0.71 & 0.71 \end{bmatrix} \cdot \begin{bmatrix} 0.00 \\ 0.00 \\ 1.04 \\ -0.54 \end{bmatrix} \cdot 10^{-4} = 7.0$$

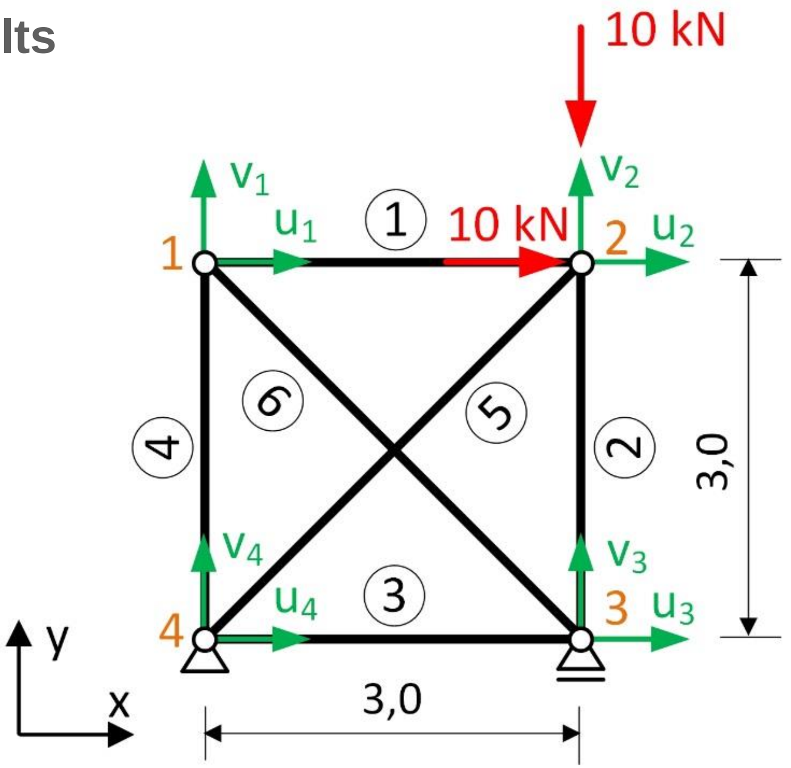
Element 6:

Element 6

$$N_6 = 1.98 \cdot 10^5 \cdot \begin{bmatrix} 0.71 & -0.71 & -0.71 & 0.71 \end{bmatrix} \cdot \begin{bmatrix} 0.18 \\ 0.00 \\ 0.86 \\ 0.18 \end{bmatrix} \cdot 10^{-4} = -7.0$$

Introductory example

Results



Nodal point displacements

Node	u [mm]	v [mm]
1	0.086	0.018
2	0.104	-0.054
3	0.018	0
4	0	0

Element forces

Element	N [kN]
1	5.0
2	-15.0
3	5.0
4	5.0
5	7.0
6	-7.0

Support forces

Node	F_x [kN]	F_y [kN]
3	-	20
4	-10	-10

FEM for truss and beam structures

Conclusions

- Elements should be appropriately connected with nodes
- Kinematic structural systems lead to unsolvable systems of equations.
  Possible program responses could be: stiffness matrix is singular, determinant is zero, program abort)
- Stiffness parameters as cross section areas, moments of inertia (for beams in bending), etc. are always to be entered in the program in order to establish the stiffness matrices.
- Support forces always fulfill the equilibirium conditions with the external loads.

Example 1

Example 2

Example 3

End

Introduction

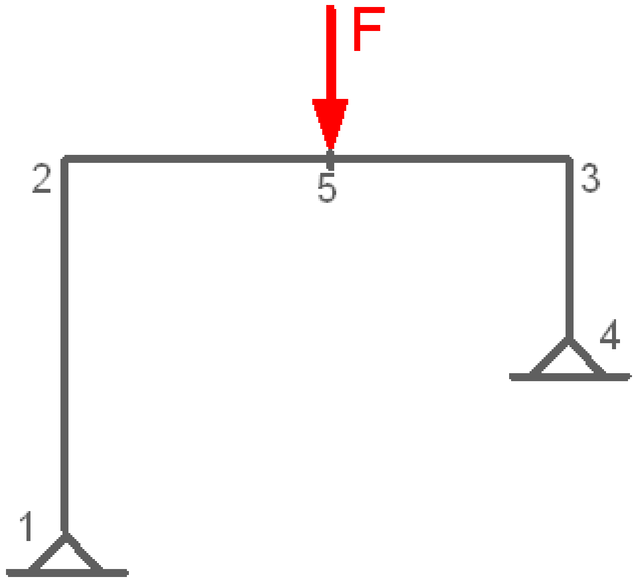
2 Truss and beam structures

Plate and shell structures

Modeling

FEM for truss and beam structures

Examples for erroneous system parameters - Example 1

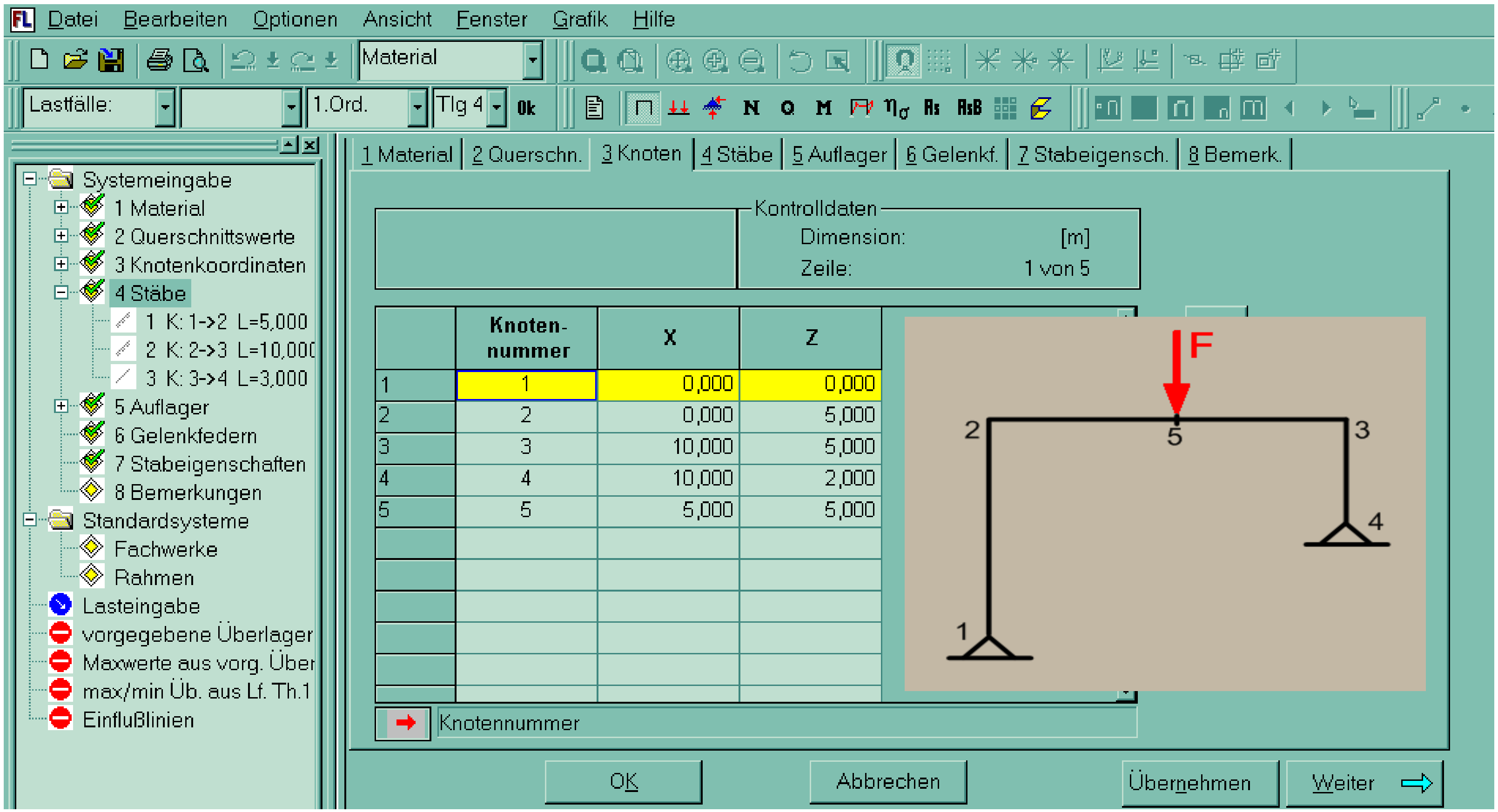


Problem:
A FE program calculates all displacements and section forces as zero, although when the input of the load is F=10 kN

How is it possible?

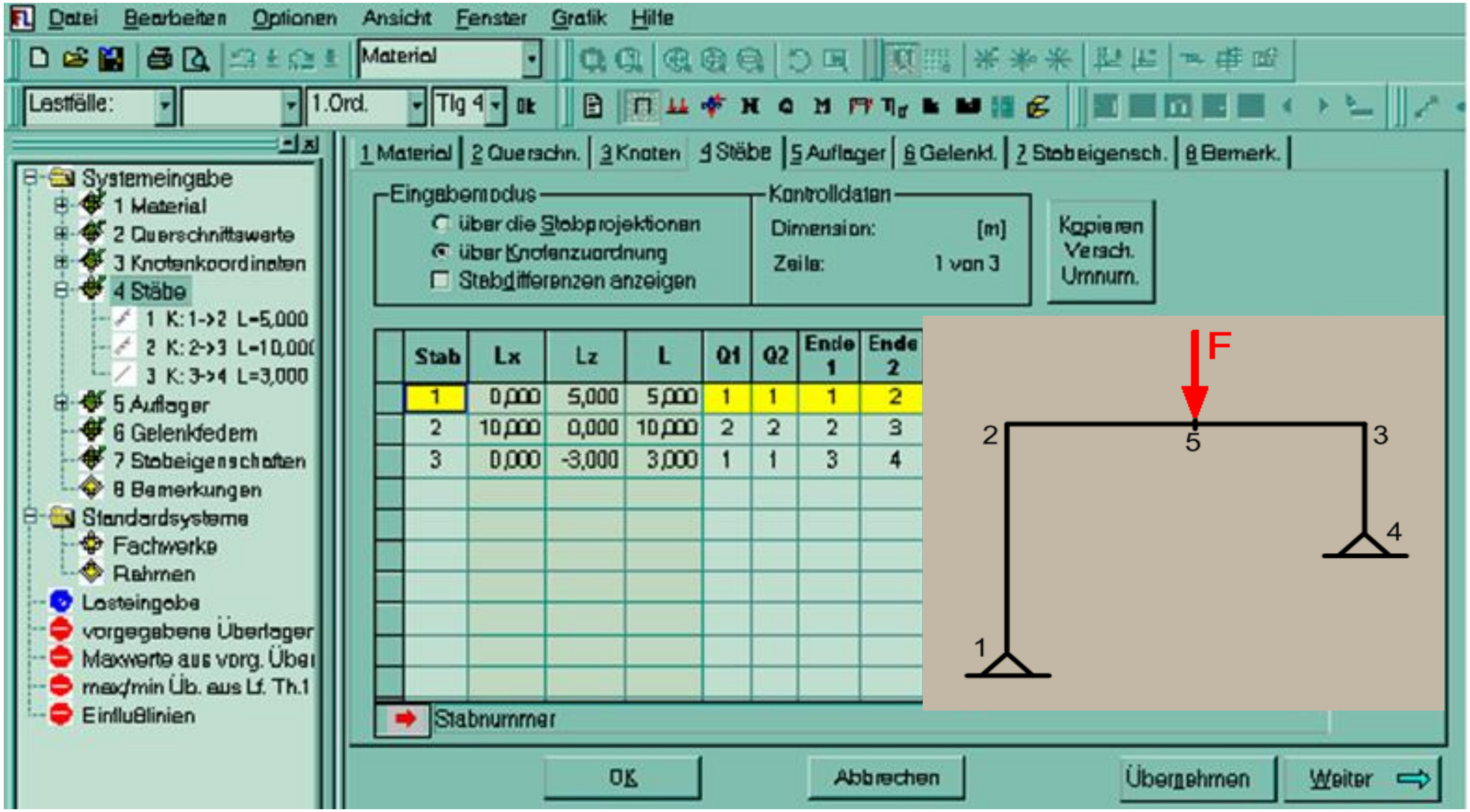
FEM for truss and beam structures

Examples for erroneous system parameters - Example 1



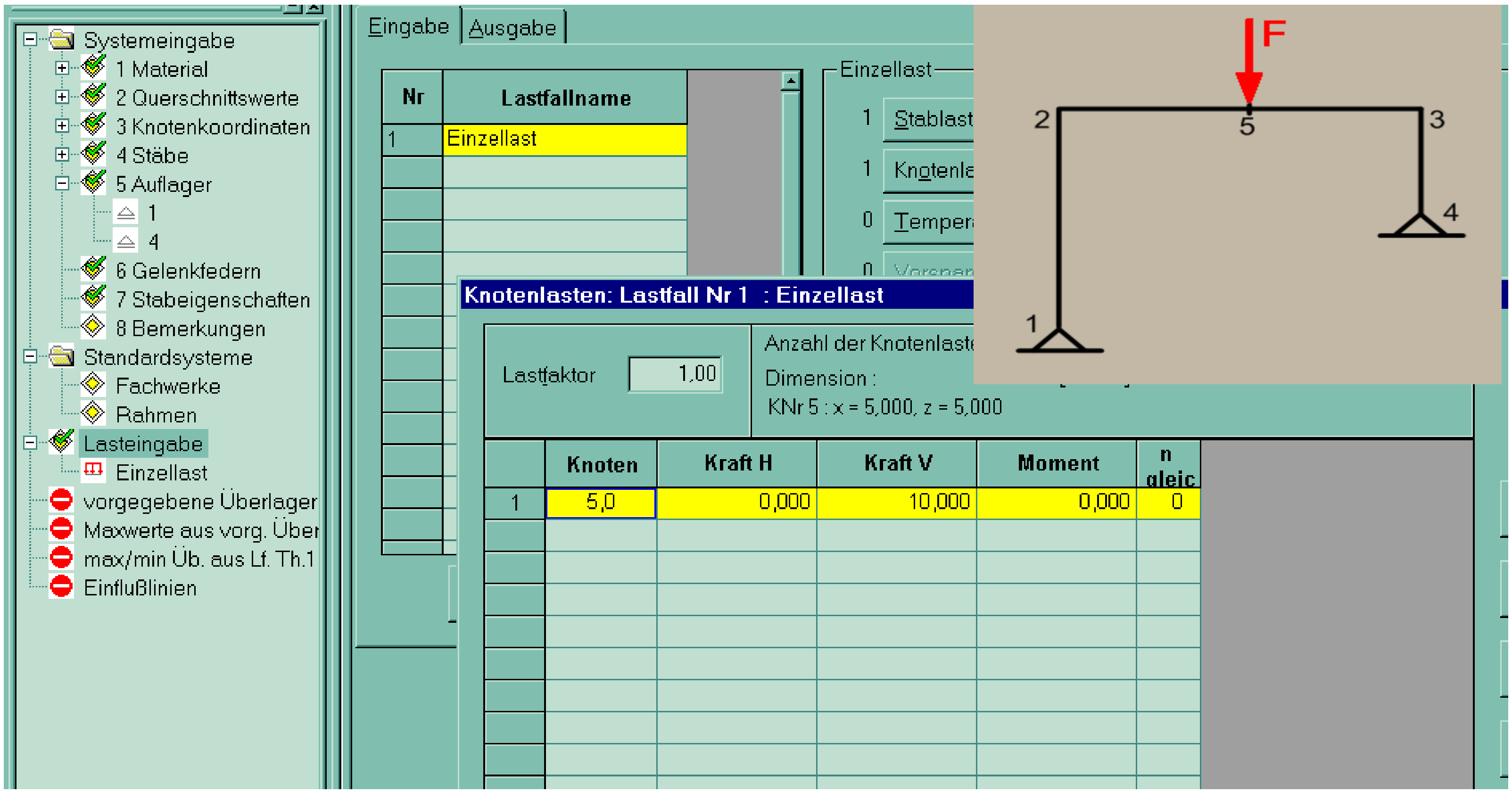
FEM for truss and beam structures

Examples for erroneous system parameters - Example 1



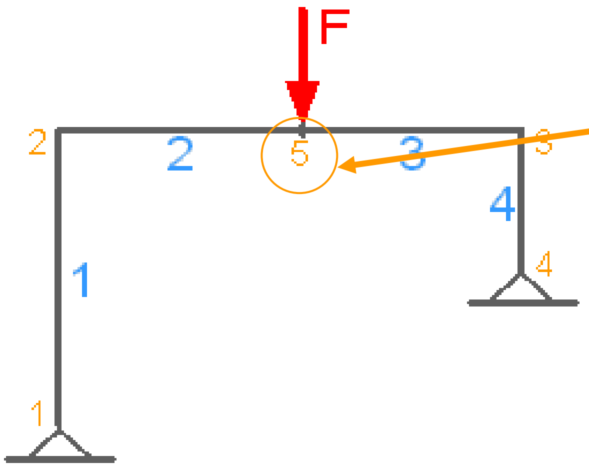
FEM for truss and beam structures

Examples for erroneous system parameters - Example 1



FEM for truss and beam structures

Examples for erroneous system parameters - Example 1



Pending node!

Error diagnostics:

Nodal point 5 has been defined and is loaded by a force F. However it is not connected to the FE system. Node 5 is a „pending node“.

Element definition (erroneous):

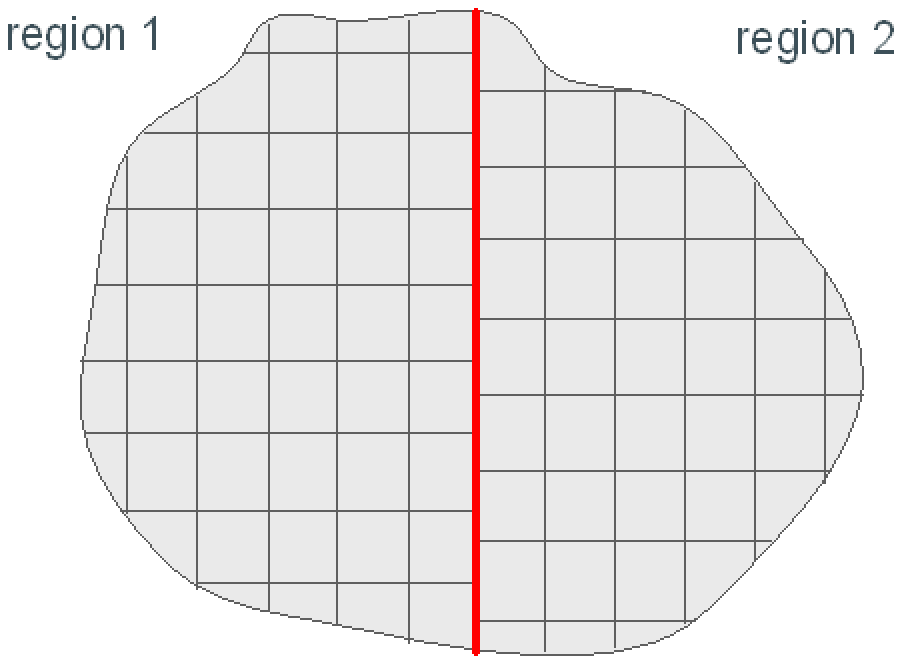
Element number	Node 1	Node 2
1	1	2
2	2	3
3	3	4

Element definition (correct):

Element number	Node 1	Node 2
1	1	2
2	2	5
3	5	3
4	3	4

FEM for truss and beam structures

Examples for erroneous system parameters - Example 2

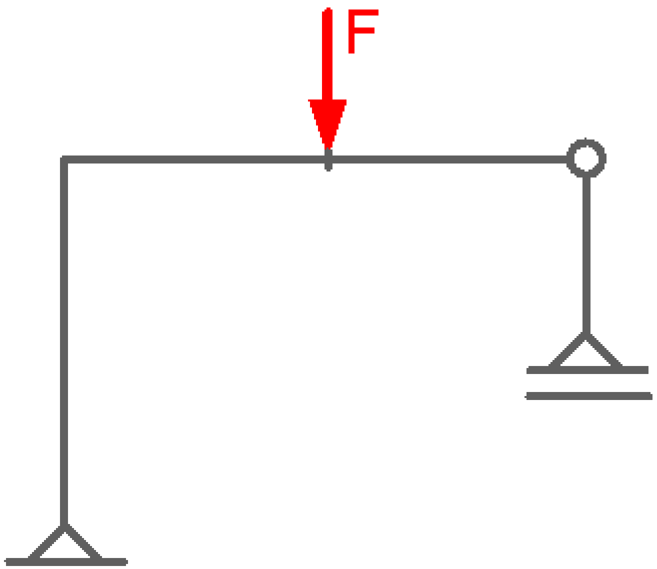


Problem

Rectangular finite elements in two adjoining regions of a plate have been generated by a FE Program.

Attention! The elements at the common interface are NOT connected unless special elements for this purpose are used. FE nets have to be checked carefully!

Examples for erroneous system parameters - Example 3



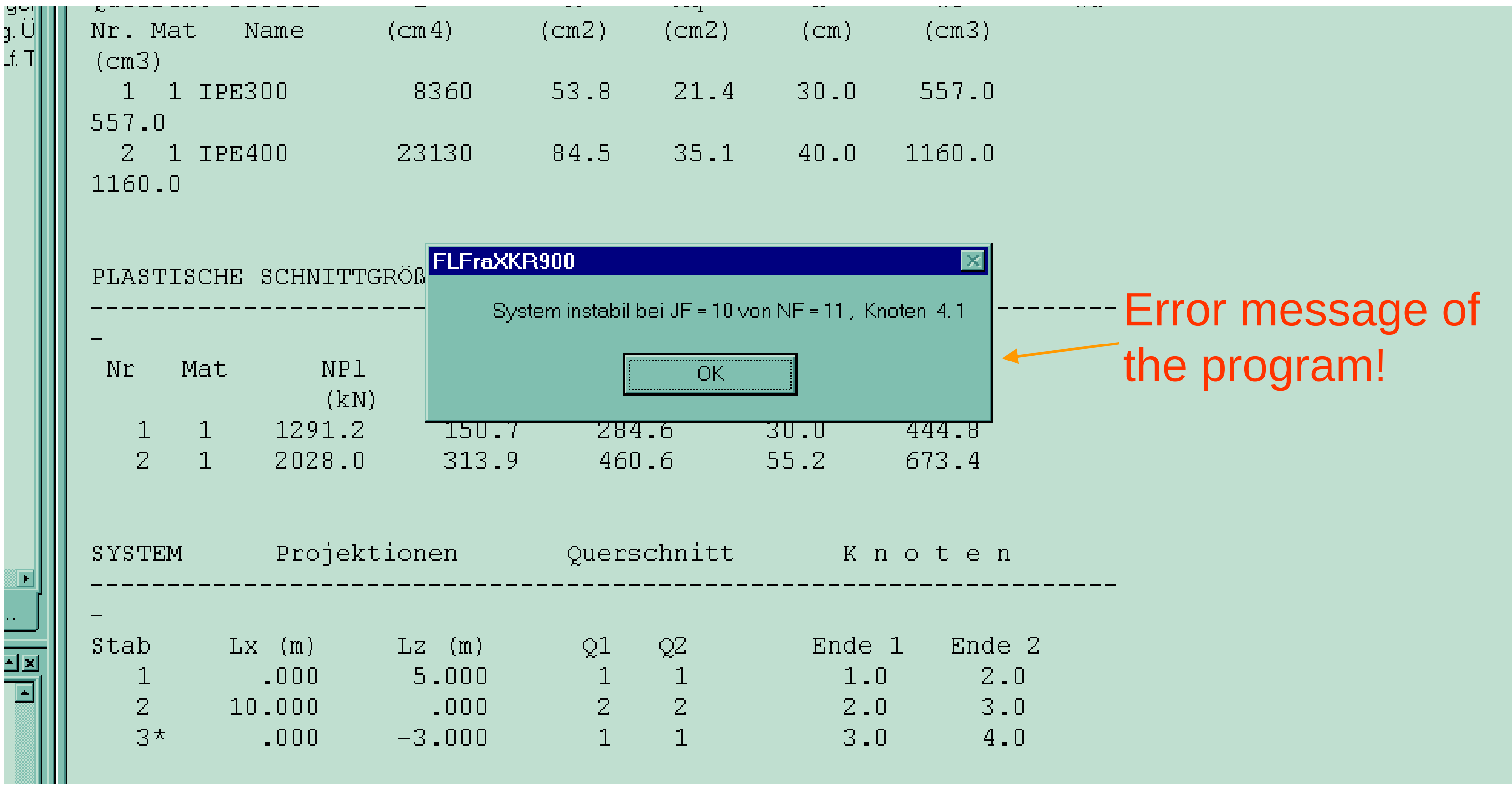
Instable system !

Problem:

A instable system is entered into a FE program. What is the response of the program?

FEM for truss and beam structures

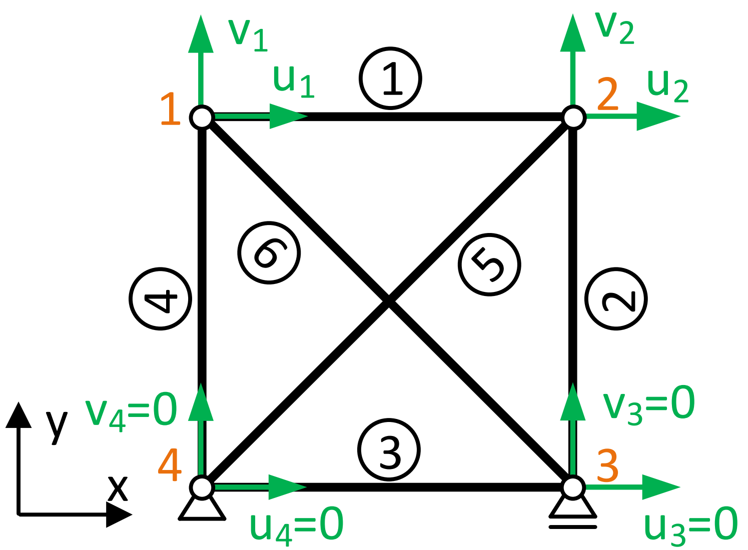
Examples for erroneous system parameters - Example 3

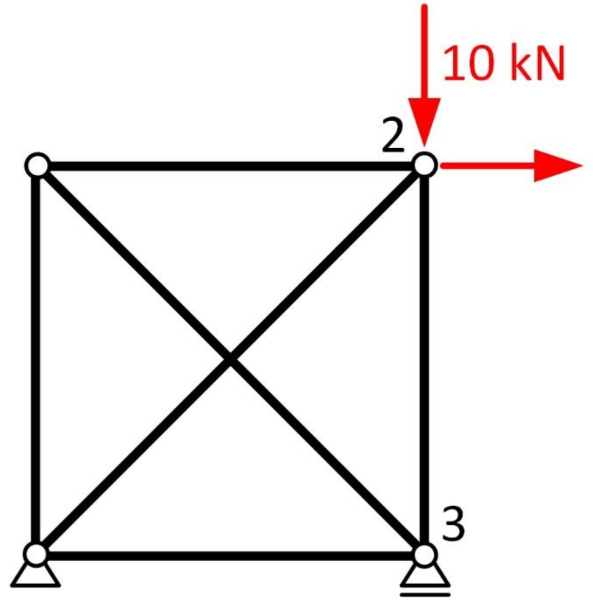


Error message of the program!

Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices





$E = 2.1 \cdot 10^8 \text{ kN/m}^2 \quad A = 0.004 \text{ m}^2$

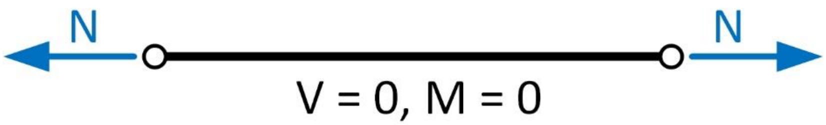
$\ell_1 - \ell_4 = 3.00 \text{m} \qquad \ell_5 - \ell_6 = 3.00 \cdot \sqrt{2} = 4.24 \text{ m}$

Element stiffness matrix of a truss element

Element stiffness matrix

$$\frac{E \cdot A}{\ell} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

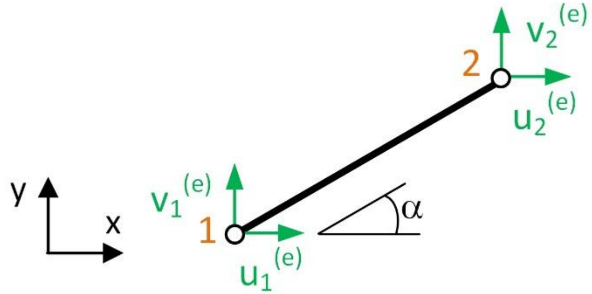
Element section force matrix

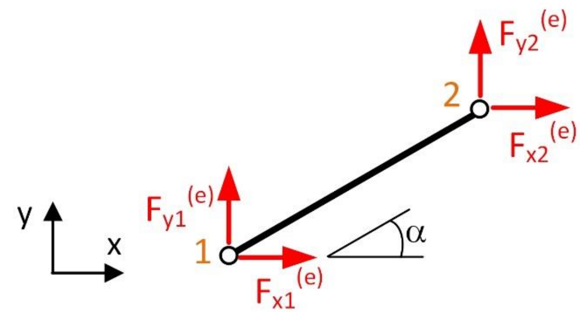


$$N = \frac{E \cdot A}{\ell} \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Element stiffness matrix of a truss element

Degrees of freedom:





Stiffness matrix:

$$\frac{E \cdot A}{\ell} \cdot \begin{bmatrix} \cos^2\alpha & \sin\alpha\cdot\cos\alpha & -\cos^2\alpha & -\sin\alpha\cdot\cos\alpha \\ \sin\alpha\cdot\cos\alpha & \sin^2\alpha & -\sin\alpha\cdot\cos\alpha & -\sin^2\alpha \\ -\cos^2\alpha & -\sin\alpha\cdot\cos\alpha & \cos^2\alpha & \sin\alpha\cdot\cos\alpha \\ -\sin\alpha\cdot\cos\alpha & -\sin^2\alpha & \sin\alpha\cdot\cos\alpha & \sin^2\alpha \end{bmatrix} \cdot \begin{bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \end{bmatrix} = \begin{bmatrix} F_{x1}^{(e)} \\ F_{y1}^{(e)} \\ F_{x2}^{(e)} \\ F_{y2}^{(e)} \end{bmatrix}$$

$$\underline{F}^{(e)} = \underline{K}^{(e)} \cdot \underline{u}^{(e)} \quad \textit{with} \quad \underline{K}^{(e)} = \underline{T}^{T} \cdot \underline{K}^{(lok)} \underline{T}$$

Section force matrix for normal force:

$$N = \frac{E \cdot A}{\ell} \cdot \begin{bmatrix} -\cos\alpha & -\sin\alpha & \cos\alpha & \sin\alpha \end{bmatrix} \cdot \begin{bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \end{bmatrix}$$

$$N = \underline{S}^{(e)} \cdot \underline{u}^{(e)}$$

Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices

Element 1: $\alpha = 0°$

$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \cdot \frac{0.004}{3.00} = 2.8 \cdot 10^5$$

Element stiffness matrix

$$\begin{bmatrix} F_{x1}^{(1)} \\ F_{x2}^{(1)} \end{bmatrix} = 2.80 \cdot 10^5 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Element section force matrices

$$N = 2.8 \cdot 10^5 \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices

Element 2: $\alpha = 90°$

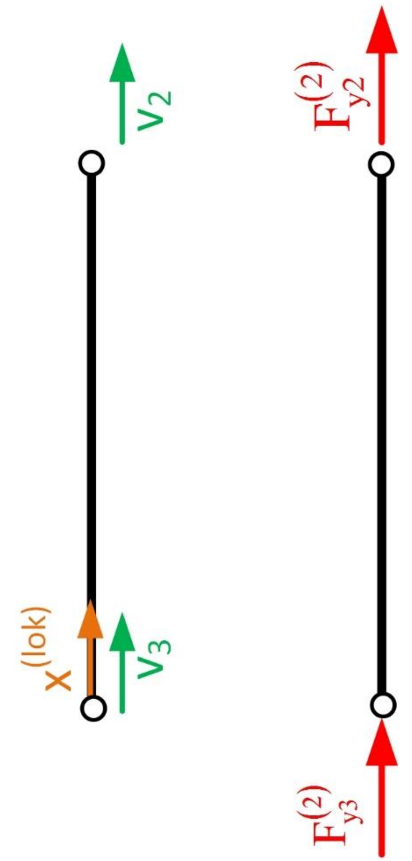
$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \cdot \frac{0.004}{3.00} = 2.8 \cdot 10^5$$

Element stiffness matrix

$$\begin{bmatrix} F_{y3}^{(2)} \\ F_{y2}^{(2)} \end{bmatrix} = 2.80 \cdot 10^5 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_3 \\ v_2 \end{bmatrix}$$

Element section force matrices

$$N = 2.8 \cdot 10^5 \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_3 \\ v_2 \end{bmatrix}$$



Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices

Element 3: $\alpha = 0°$

$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \cdot \frac{0.004}{3.00} = 2.8 \cdot 10^5$$

Element stiffness matrix

$$\begin{bmatrix} F_{x4}^{(3)} \\ F_{x3}^{(3)} \end{bmatrix} = 2.80 \cdot 10^5 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_4 \\ u_3 \end{bmatrix}$$

Element section force matrices

$$N = \quad 2.8 \cdot 10^5 \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_4 \\ u_3 \end{bmatrix}$$

Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices

Element 4: $\alpha = 90°$

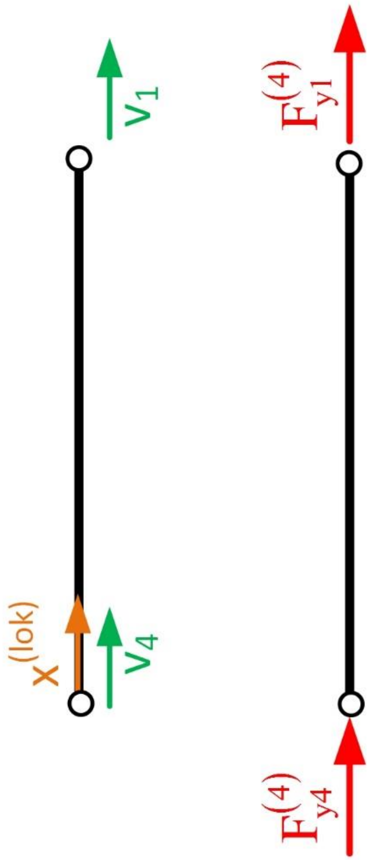
$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \cdot \frac{0.004}{3.00} = 2.8 \cdot 10^5$$

Element stiffness matrix

$$\begin{bmatrix} F_{y4}^{(4)} \\ F_{y1}^{(4)} \end{bmatrix} = 2.80 \cdot 10^5 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_4 \\ v_1 \end{bmatrix}$$

Element section force matrices

$$N = 2.8 \cdot 10^5 \cdot \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_4 \\ v_1 \end{bmatrix}$$

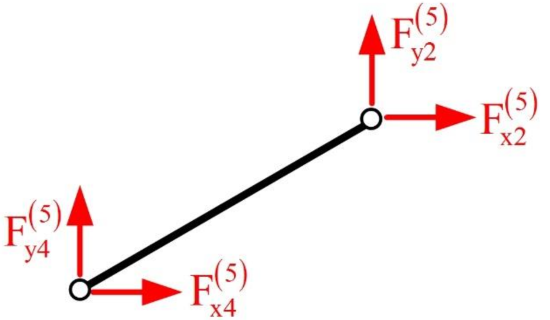


Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices

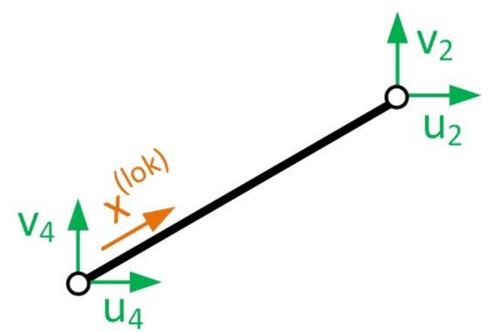
Element 5: $\alpha = 45°$

$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \, \frac{0.004}{4.24} = 1.98 \cdot 10^5$$



Element stiffness matrix

$$\begin{bmatrix} F_{x4}^{(5)} \\ F_{y4}^{(5)} \\ F_{x2}^{(5)} \\ F_{y2}^{(5)} \end{bmatrix} = 1.98\,10^5 \cdot \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} u_4 \\ v_4 \\ u_2 \\ v_2 \end{bmatrix}$$



Element section force matrices

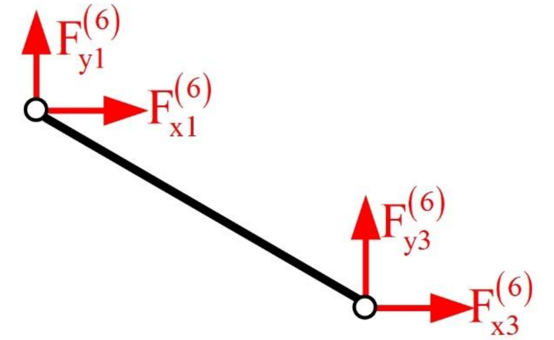
$$N = 1.98 \cdot 10^5 \cdot \begin{bmatrix} -0.707 & -0.707 & 0.707 & 0.707 \end{bmatrix} \cdot \begin{bmatrix} u_4 \\ v_4 \\ u_2 \\ v_2 \end{bmatrix}$$

Element stiffness matrix of a truss element

Example: Element stiffness and section force matrices

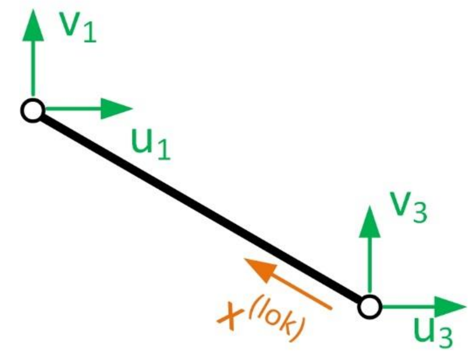
Element 6: $\alpha = 135°$

$$\frac{E \cdot A}{\ell} = 2.1 \cdot 10^8 \, \frac{0.004}{4.24} = 1.98 \cdot 10^5$$



Element stiffness matrix

$$\begin{bmatrix} F_{x3}^{(6)} \\ F_{y3}^{(6)} \\ F_{x1}^{(6)} \\ F_{y1}^{(6)} \end{bmatrix} = 1.98 \cdot 10^5 \cdot \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} u_3 \\ v_3 \\ u_1 \\ v_1 \end{bmatrix}$$



Element section force matrices

$$N = 1.98 \cdot 10^5 \cdot \begin{bmatrix} 0.707 & -0.707 & -0.707 & 0.707 \end{bmatrix} \cdot \begin{bmatrix} u_3 \\ v_3 \\ u_1 \\ v_1 \end{bmatrix}$$