Finite Elements in Structural Analysis

Introduction

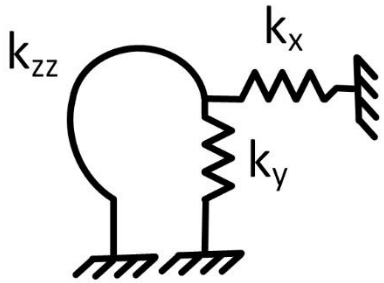
2 Truss and beam structures

Plate and shell structures

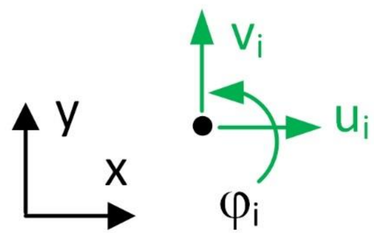
Modeling

Stiffness matrix

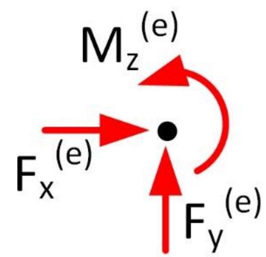
Spring elements



Springs



Displacements



external forces

$$k_x \ [kN/m]$$
$$k_y \ [kN/m]$$
$$k_{zz} \ [kN \cdot m]$$

Spring constants

Spring equations

$$F_x^{(e)} = k_x \cdot u_i$$

$$F_y^{(e)} = k_y \cdot v_i$$

$$M_z^{(e)} = k_{zz} \cdot \varphi_i$$

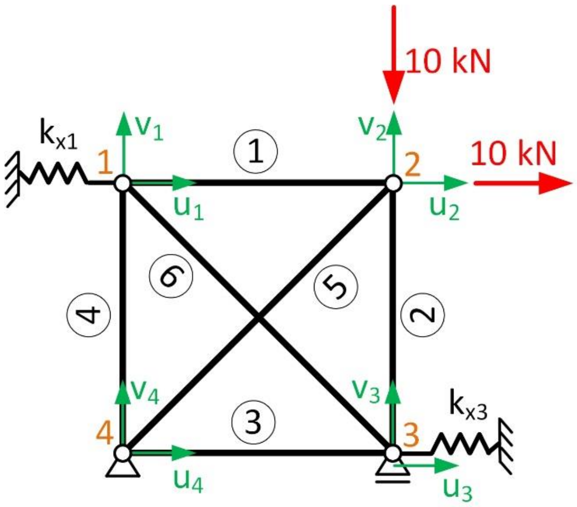
$\rightarrow$

Stiffness matrix

$$\begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_{zz} \end{bmatrix} \cdot \begin{bmatrix} u_i \\ v_i \\ \varphi_i \end{bmatrix} = \begin{bmatrix} F_x^{(e)} \\ F_y^{(e)} \\ M_z^{(e)} \end{bmatrix}$$

Example 1

Modification of the stiffness matrix by springs k_x at nodal points 1 and 3



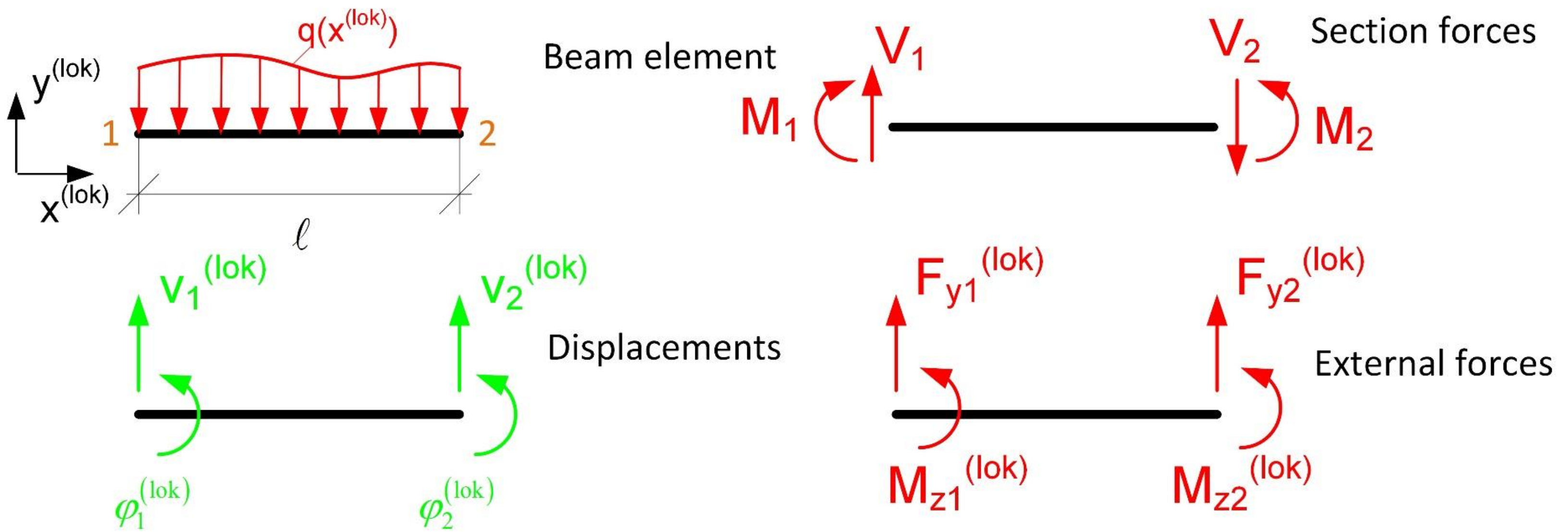
Spring k_x at nodal point 1 $\bar{k}_{x1} = k_{x1} / 2.8 \cdot 10^5$

Spring k_x at nodal point 3 $\bar{k}_{x3} = k_{x3} / 2.8 \cdot 10^5$

$$
\begin{array}{cc}
 & \begin{array}{ccccc} u_1 & v_1 & u_2 & v_2 & u_3 \end{array} \\
2.8 \cdot 10^5 \cdot & \begin{bmatrix} 1.35 + \bar{k}_{x1} & -0.35 & -1.0 & 0 & -0.35 \\ -0.35 & 1.35 & 0 & 0 & 0.35 \\ -1.0 & 0 & 1.35 & 0.35 & 0 \\ 0 & 0 & 0.35 & 1.35 & 0 \\ -0.35 & 0.35 & 0 & 0 & 1.35 + \bar{k}_{x3} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \\ -10 \\ 0 \end{bmatrix}
\end{array}
$$

Stiffness and section forces matrix of a beam

Beam element with element load



Signs: positive in the direction of positive local coordinates

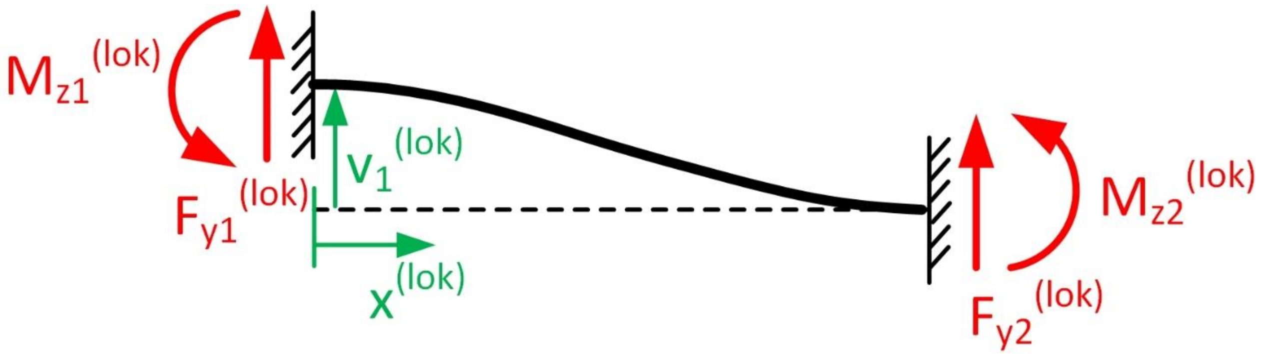
Deriviation of the stiffness matrix

$F_{y1}^{(lok)}, F_{y2}^{(lok)}, M_1^{(lok)}, M_2^{(lok)}$ are the restraint forces and moments, resulting from displacements and rotations $V_1^{(lok)}, \varphi_1^{(lok)}, V_2^{(lok)}, \varphi_2^{(lok)}$ as well as from the element load $q(x^{(lok)})$.

Stiffness and section forces matrix of a beam

Derivation of the stiffness matrix

Element forces and moments due to the displacement of nodal point 1



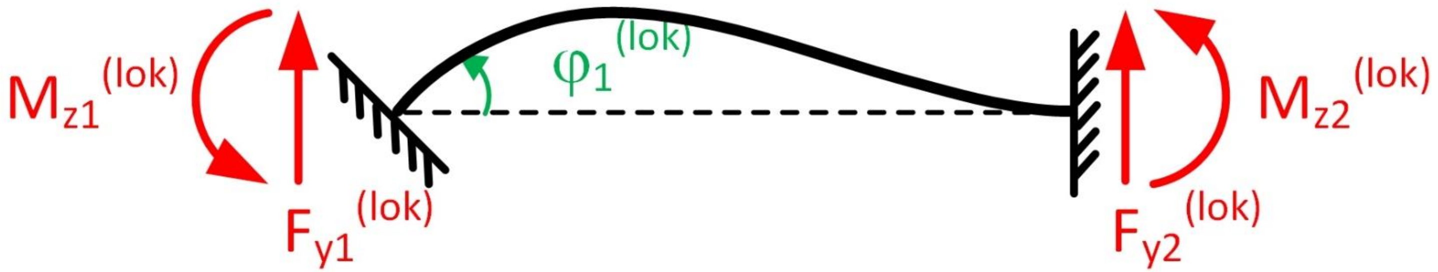
$$F_{y1}^{(lok)} = 12 \cdot EI / \ell^3 \cdot v_1^{(lok)} \qquad F_{y2}^{(lok)} = -12 \cdot EI / \ell^3 \cdot v_1^{(lok)}$$

$$M_{z1}^{(lok)} = 6 \cdot EI / \ell^2 \cdot v_1^{(lok)} \qquad M_{z2}^{(lok)} = 6 \cdot EI / \ell^2 \cdot v_1^{(lok)}$$

Stiffness and section forces matrix of a beam

Derivation of the stiffness matrix

Element forces and moments due to the rotation of nodal point 1



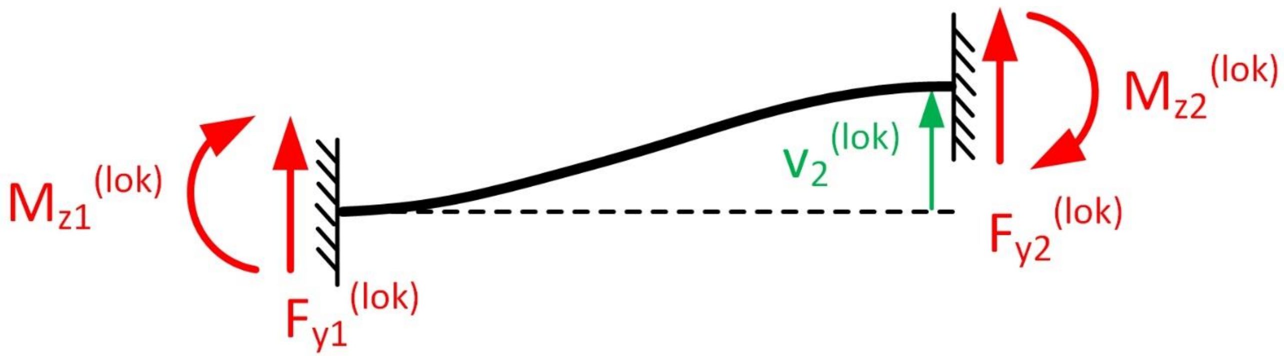
$$F_{y1}^{(lok)} = 6 \cdot EI / \ell^2 \cdot \varphi_1^{(lok)} \qquad F_{y2}^{(lok)} = -6 \cdot EI / \ell^2 \cdot \varphi_1^{(lok)}$$

$$M_{z1}^{(lok)} = 4 \cdot EI / \ell \cdot \varphi_1^{(lok)} \qquad M_{z2}^{(lok)} = 2 \cdot EI / \ell \cdot \varphi_1^{(lok)}$$

Stiffness and section forces matrix of a beam

Derivation of the stiffness matrix

Element forces and moments due to the displacement of nodal point 2



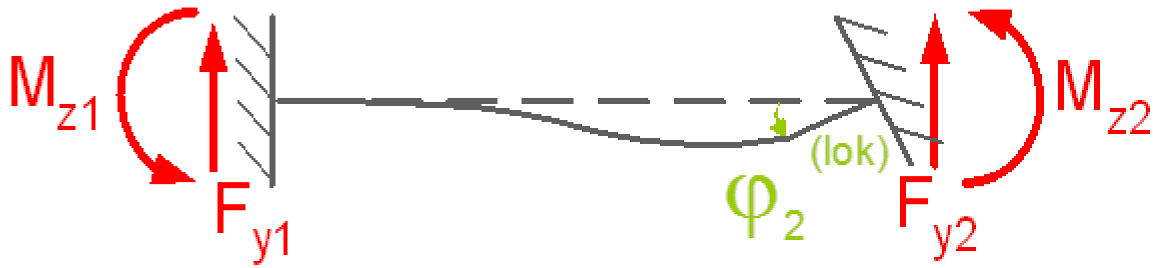
$$F_{y1}^{(lok)} = -12 \cdot EI/\ell^3 \cdot v_2^{(lok)} \qquad F_{y2}^{(lok)} = 12 \cdot EI/\ell^3 \cdot v_2^{(lok)}$$

$$M_{z1}^{(lok)} = -6 \cdot EI/\ell^2 \cdot v_2^{(lok)} \qquad M_{z2}^{(lok)} = -6 \cdot EI/\ell^2 \cdot v_2^{(lok)}$$

Stiffness and section forces matrix of a beam

Derivation of the stiffness matrix

Element forces and moments due to the rotation of nodal point 2



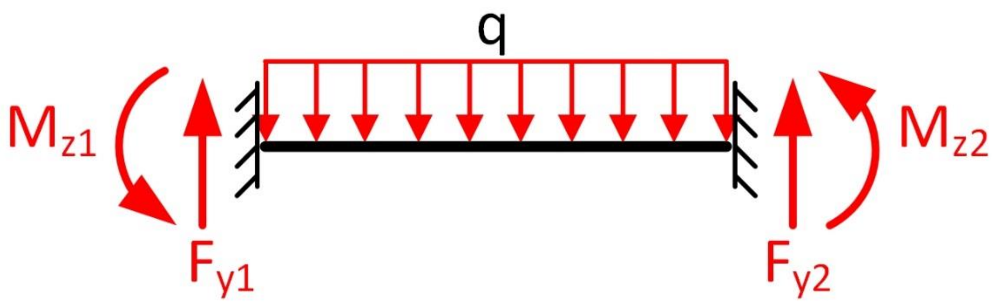
$$F_{y1}^{(lok)} = 6 \cdot EI / \ell^2 \cdot \varphi_2^{(lok)} \qquad F_{y2}^{(lok)} = -6 \cdot EI / \ell^2 \cdot \varphi_2^{(lok)}$$

$$M_{z1}^{(lok)} = 2 \cdot EI / \ell \cdot \varphi_2^{(lok)} \qquad M_{z2}^{(lok)} = 4 \cdot EI / \ell \cdot \varphi_2^{(lok)}$$

Stiffness and section forces matrix of a beam

Derivation of the stiffness matrix

Element forces and moments due to element loads



$$F_{y1}^{(lok)} = q \cdot \ell / 2 \qquad F_{y2}^{(lok)} = q \cdot \ell / 2$$

$$M_{z1}^{(lok)} = q \cdot \ell^2 / 12 \qquad M_{z2}^{(lok)} = -q \cdot \ell^2 / 12$$

Stiffness and section forces matrix of a beam

$$e.g.\quad F_{y1}^{(lok)} = \frac{12\ E\cdot I}{\ell^3}\cdot v_1^{(lok)} + \frac{6\ E\cdot I}{\ell^2}\cdot \varphi_1^{(lok)} - \frac{12\ E\cdot I}{\ell^3}\cdot v_2^{(lok)} + \frac{6\ E\cdot I}{\ell^2}\cdot \varphi_2^{(lok)} + F_L^{\,1}$$

The terms for the nodal point displacements and rotations lead to the stiffness matrix

The restraint forces and moments due to **q** lead to the element load vector

$$\frac{E\cdot I}{\ell}\cdot\begin{bmatrix} 12/\ell^2 & 6/\ell & -12/\ell^2 & 6/\ell \\ 6/\ell & 4 & -6/\ell & 2 \\ -12/\ell^2 & -6/\ell & 12/\ell^2 & -6/\ell \\ 6/\ell & 2 & -6/\ell & 4 \end{bmatrix}\cdot\begin{bmatrix} v_1^{(lok)} \\ \varphi_1^{(lok)} \\ v_2^{(lok)} \\ \varphi_2^{(lok)} \end{bmatrix} = \begin{bmatrix} F_{y1}^{(lok)} \\ M_{z1}^{(lok)} \\ F_{y2}^{(lok)} \\ M_{z2}^{(lok)} \end{bmatrix} - \begin{bmatrix} F_{L1} \\ M_{L1} \\ F_{L2} \\ M_{L2} \end{bmatrix}$$

$$\underline{K}_{be}^{(lok)}\cdot\underline{u}_{be}^{(lok)} = \underline{F}_{be}^{(lok)} - \underline{F}_{bL}^{(lok)}$$

Stiffness and section forces matrix of a beam

Section forces matrix

Stiffness matrix

$$\begin{bmatrix} F_{y1}^{(lok)} \\ M_{z1}^{(lok)} \\ F_{y2}^{(lok)} \\ M_{z2}^{(lok)} \end{bmatrix} = \frac{E \cdot I}{\ell} \cdot \begin{bmatrix} 12/\ell^2 & 6/\ell & -12/\ell^2 & 6/\ell \\ 6/\ell & 4 & -6/\ell & 2 \\ -12/\ell^2 & -6/\ell & 12/\ell^2 & -6/\ell \\ 6/\ell & 2 & -6/\ell & 4 \end{bmatrix} \cdot \begin{bmatrix} v_1^{(lok)} \\ \varphi_1^{(lok)} \\ v_2^{(lok)} \\ \varphi_2^{(lok)} \end{bmatrix} + \begin{bmatrix} F_{L1} \\ M_{L1} \\ F_{L2} \\ M_{L2} \end{bmatrix}$$

The section forces matrix is obtained from t:he stiffness matrix with:

$$V_1 = F_{y1}^{(lok)}$$

$$M_1 = -M_{z1}^{(lok)}$$

$$V_2 = -F_{y2}^{(lok)}$$

$$M_2 = M_{z2}^{(lok)}$$

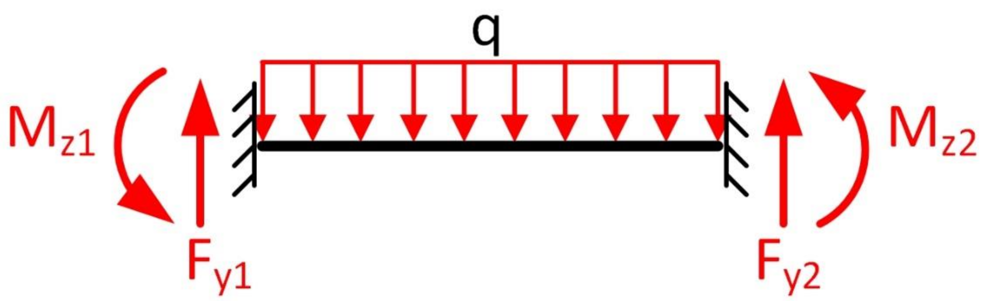
Section forces matrix

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \frac{E \cdot I}{\ell} \cdot \begin{bmatrix} 12/\ell^2 & 6/\ell & -12/\ell^2 & 6/\ell \\ -6/\ell & -4 & 6/\ell & -2 \\ 12/\ell^2 & 6/\ell & -12/\ell^2 & 6/\ell \\ 6/\ell & 2 & -6/\ell & 4 \end{bmatrix} \cdot \begin{bmatrix} v_1^{(lok)} \\ \varphi_1^{(lok)} \\ v_2^{(lok)} \\ \varphi_2^{(lok)} \end{bmatrix} + \begin{bmatrix} F_{L1} \\ -M_{L1} \\ -F_{L2} \\ M_{L2} \end{bmatrix}$$

$$\underline{S} = \underline{S}_{be}^{(lok)} \cdot \underline{u}_{be}^{(lok)} + \underline{F}_{LS}^{(lok)}$$

Stiffness and section forces matrix of a beam

Element load vector for uniformly distributed load



$$F_{y1}^{(lok)} = q \cdot \ell / 2$$

$$M_{z1}^{(lok)} = q \cdot \ell^2 / 12$$

$$F_{y2}^{(lok)} = q \cdot \ell / 2$$

$$M_{z2}^{(lok)} = -q \cdot \ell^2 / 12$$

$$\begin{bmatrix} F_{L1} \\ M_{L1} \\ F_{L2} \\ M_{L2} \end{bmatrix} = \begin{bmatrix} q \cdot \ell / 2 \\ q \cdot \ell^2 / 12 \\ q \cdot \ell / 2 \\ -q \cdot \ell^2 / 12 \end{bmatrix}$$

Stiffness and section forces matrix of a beam

Stiffness matrix

$$\frac{E\cdot I}{\ell}\cdot\begin{bmatrix} 12/\ell^2 & 6/\ell & -12/\ell^2 & 6/\ell \\ 6/\ell & 4 & -6/\ell & 2 \\ -12/\ell^2 & -6/\ell & 12/\ell^2 & -6/\ell \\ 6/\ell & 2 & -6/\ell & 4 \end{bmatrix}\cdot\begin{bmatrix} v_1^{(lok)} \\ \varphi_1^{(lok)} \\ v_2^{(lok)} \\ \varphi_2^{(lok)} \end{bmatrix}=\begin{bmatrix} F_{y1}^{(lok)} \\ M_{z1}^{(lok)} \\ F_{y2}^{(lok)} \\ M_{z2}^{(lok)} \end{bmatrix}$$

$$\underline{K}_{be}^{(lok)}\cdot\underline{u}_{be}^{(lok)}=\underline{F}_{be}^{(lok)}$$

Section forces matrix

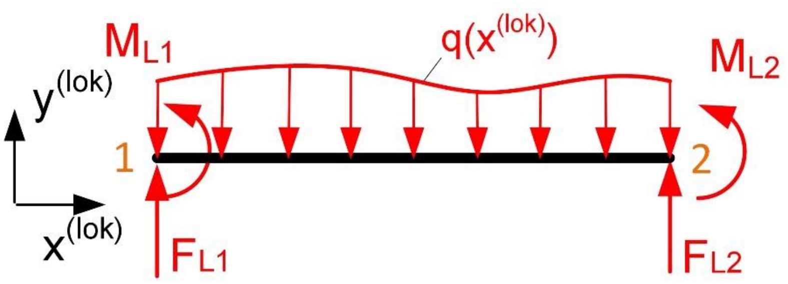
$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix}=\frac{E\cdot I}{\ell}\cdot\begin{bmatrix} 12/\ell^2 & 6/\ell & -12/\ell^2 & 6/\ell \\ -6/\ell & -4 & 6/\ell & -2 \\ 12/\ell^2 & 6/\ell & -12/\ell^2 & 6/\ell \\ 6/\ell & 2 & -6/\ell & 4 \end{bmatrix}\cdot\begin{bmatrix} v_1^{(lok)} \\ \varphi_1^{(lok)} \\ v_2^{(lok)} \\ \varphi_2^{(lok)} \end{bmatrix}$$

$$\underline{S}=\underline{S}_{be}^{(lok)}\cdot\underline{u}_{be}^{(lok)}$$

Element loads of a beam

Consideration of element loads:

- For the computation of the global system the element loads will be replaced by nodal forces and moments (equivalent nodal loads) $F_{L1}, M_{L1}, F_{L2}, M_{L2}$.
- The equivalent loads are the support reactions of the fixed beam, applied with opposite sign on the global system.



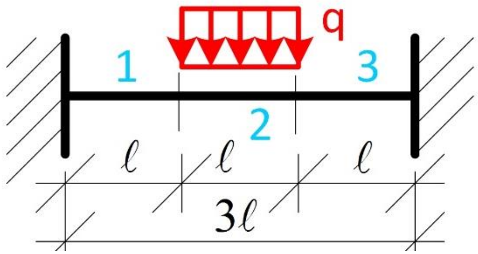
Positive direction of the support reactions

Element

Element loads of a beam

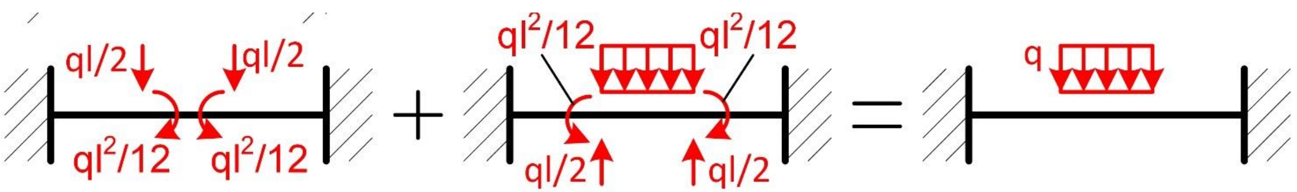
Example 2: Element loads on a simple beam system

System:



For a global system with three beam elements the section forces are determined by means of equivalent nodel point loads.

Equivalent nodal loads:



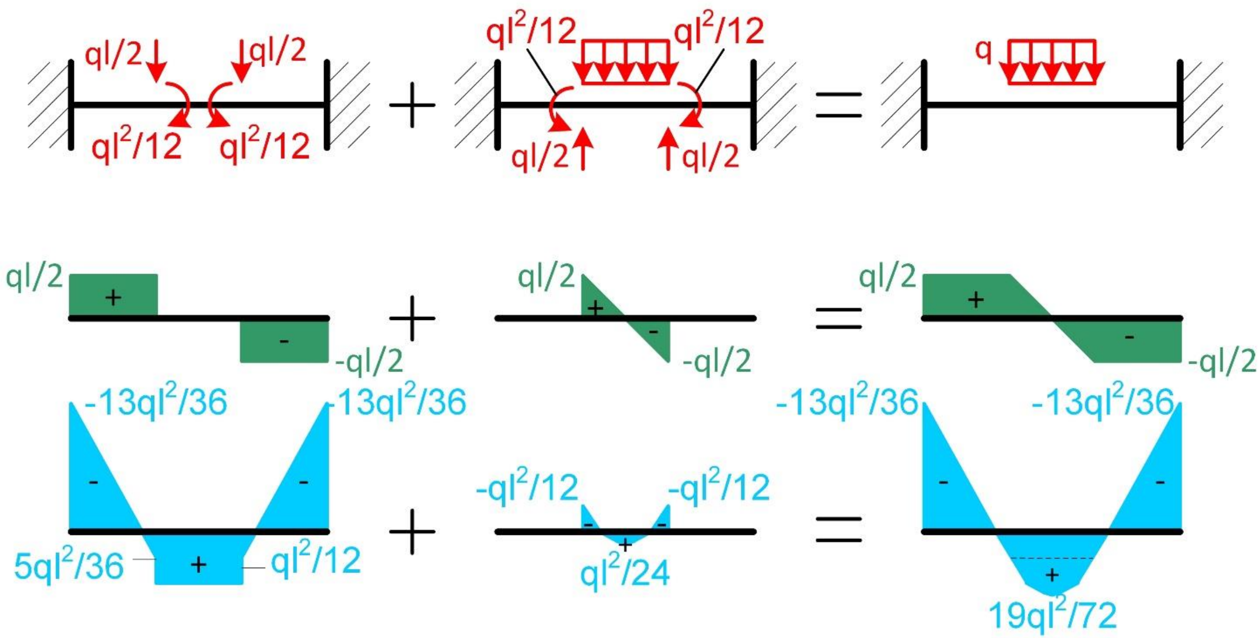
Element loads of a beam

Example 2: Element loads on a simple beam system

Superposition

Shear forces

Moments



System with equivalent loads

Beam with element load fixed on both sides

Superposition of the section forces

Element loads of a beam

Consideration of element loads with equivalent nodal loads

1. Determination of restraining forces and moments of the loaded element fixed at both ends.

2. Application of the restraining forces and moments with opposite sign on the global structural system as nodal loads (equivalent nodal loads).

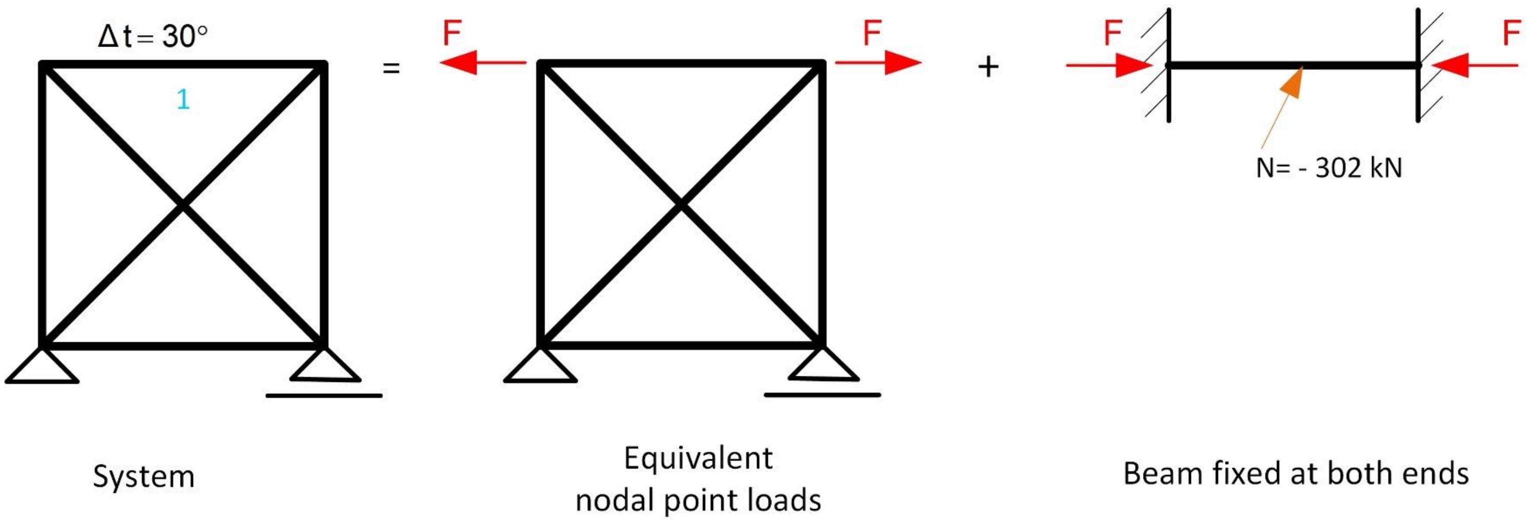
3. Computation of the global system with the equivalent nodal loads.

4. Superposition of the section forces due to the element load on the restrained element with the section forces of the global structural system.

Element loads of a beam

Example 3: Temperature loading

Element 1 is heated up to 30°C. The normal forces in the elements have to be determined.



Equivalent nodal forces

$$F = E \cdot A \cdot \alpha_T \cdot \Delta_t \qquad F = 2.1 \cdot 10^8 \cdot 0.004 \cdot 1.2 \cdot 10^{-5} \cdot 30 = 302 \text{ kN} \qquad N = -302 \text{ kN}$$

Element loads of a beam

Example 3: Temperature loading

System of equations:

$$2.80\cdot 10^5 \begin{bmatrix} 1.35 & -0.35 & -1. & 0 & -0.35 \\ -0.35 & 1.35 & 0 & 0 & 0.35 \\ -1. & 0 & 1.35 & 0.35 & 0 \\ 0 & 0 & 0.35 & 1.35 & 0 \\ -0.35 & 0.35 & 0 & 0 & 1.35 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -302 \\ 0 \\ 302 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -0.540 \\ -0.112 \\ 0.428 \\ -0.112 \\ 0.112 \end{bmatrix} \cdot 10^{-3}\ m$$

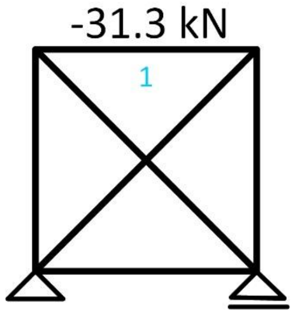
Normal forces:

$$N_1 = 271.1\,kN \qquad N_2 = -31.3\,kN \qquad N_3 = -31.3\,kN$$

$$N_4 = 5.0\,kN \qquad N_5 = 44.3\,kN \qquad N_6 = 44.3\,kN$$

Superposition:

$$N_1 = 271.1 - 302.4 = -31.3\,kN$$



Beams in bending with longitudinal and shear stiffness

Extension for normal forces and shear stiffness

- Extension by normal forces is required for the analysis of frames etc.

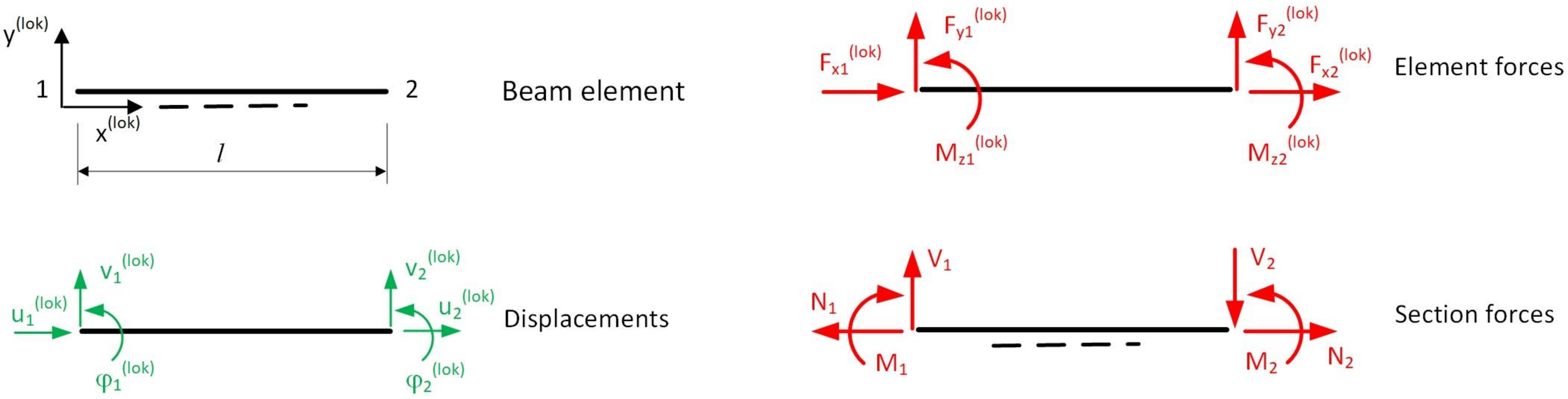
- Shear stiffness is required if shear deformations are significant; they are normally included in beam elements implemented in FE programs.

Extension of the stiffness matrix

- *Longitudinal stiffness:* The stiffness matrix of the bending terms of the beam will be extended by the entries of the truss element.
- *Shear stiffness:* Solution of the differential equation for the beam with shear deformation.

Stiffness and section forces matrix of a beam

Beam element



Beams in bending with longitudinal and shear stiffness

Stiffness matrix

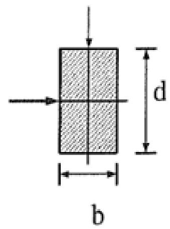
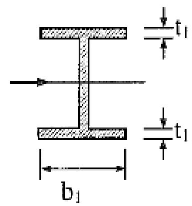
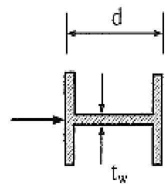
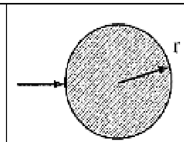
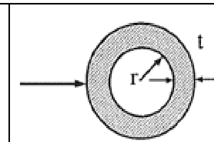
$$\begin{bmatrix} a_0 & 0 & 0 & -a_0 & 0 & 0 \\ 0 & \frac{12 a_1}{\ell^2} & \frac{6 a_1}{\ell} & 0 & \frac{-12 a_1}{\ell^2} & \frac{6 a_1}{\ell} \\ 0 & \frac{6 a_1}{\ell} & 4 a_2 & 0 & \frac{-6 a_1}{\ell} & 2 a_3 \\ -a_0 & 0 & 0 & a_0 & 0 & 0 \\ 0 & \frac{-12 a_1}{\ell^2} & \frac{-6 a_1}{\ell} & 0 & \frac{12 a_1}{\ell^2} & \frac{-6 a_1}{\ell} \\ 0 & \frac{6 a_1}{\ell} & 2 a_3 & 0 & \frac{-6 a_1}{\ell} & 4 a_2 \end{bmatrix} \cdot \begin{bmatrix} u_1^{(lok)} \\ v_1^{(lok)} \\ \varphi_1^{(lok)} \\ u_2^{(lok)} \\ v_2^{(lok)} \\ \varphi_2^{(lok)} \end{bmatrix} = \begin{bmatrix} F_{x1}^{(lok)} \\ F_{y1}^{(lok)} \\ M_{z1}^{(lok)} \\ F_{x2}^{(lok)} \\ F_{y2}^{(lok)} \\ M_{z2}^{(lok)} \end{bmatrix}$$

with $a_0 = \frac{E \cdot A}{\ell}$ $\quad a_1 = \frac{E \cdot I}{\ell \cdot (1+m)}$ $\quad a_2 = \frac{E \cdot I \cdot (4+m)}{4 \cdot \ell \cdot (1+m)}$ $\quad a_3 = \frac{E \cdot I \cdot (2-m)}{2 \cdot \ell \cdot (1+m)}$ $\quad m = \frac{12 \cdot E \cdot I}{G \cdot A_s \cdot \ell^2}$

Beams in bending with longitudinal and shear stiffness

Practical hints

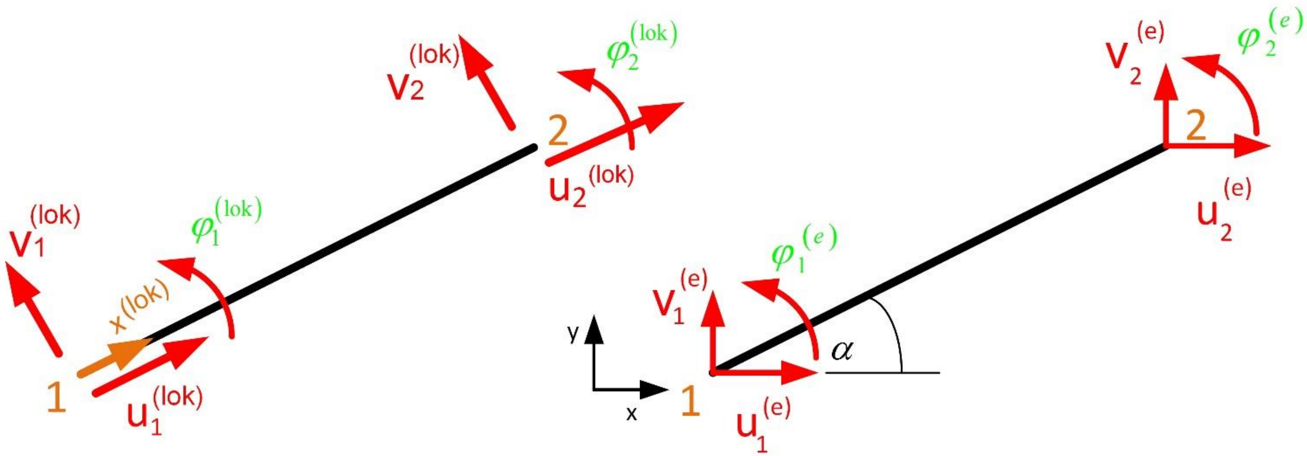
- Shear areas for some cross sections:

Cross section					
Shear area	$\frac{5}{6}\cdot b\cdot d$	$\frac{5}{3}\cdot t_1\cdot b_1$	$t_W\cdot d$	$0{,}9\cdot\pi\cdot r^2$	$\pi\cdot r\cdot t$

- Shear deformations can be excluded by defining a huge value for the shear area A_s , e.g. A_s = 1000·A .
- If shear area is set to zero with $A_s = 0$ kinematic mechanisms may occur.
- Some programs define the input $A_s = 0$ in order to neglect the shear area. In the program the mechanically correct shear area is set as „infinite“.

Coordinate transformation

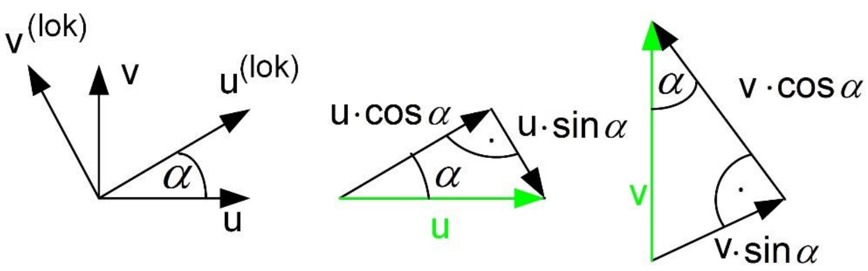
Transformation of the nodal point displacements



Local coordinates

global coordinates

Coordinate transformation



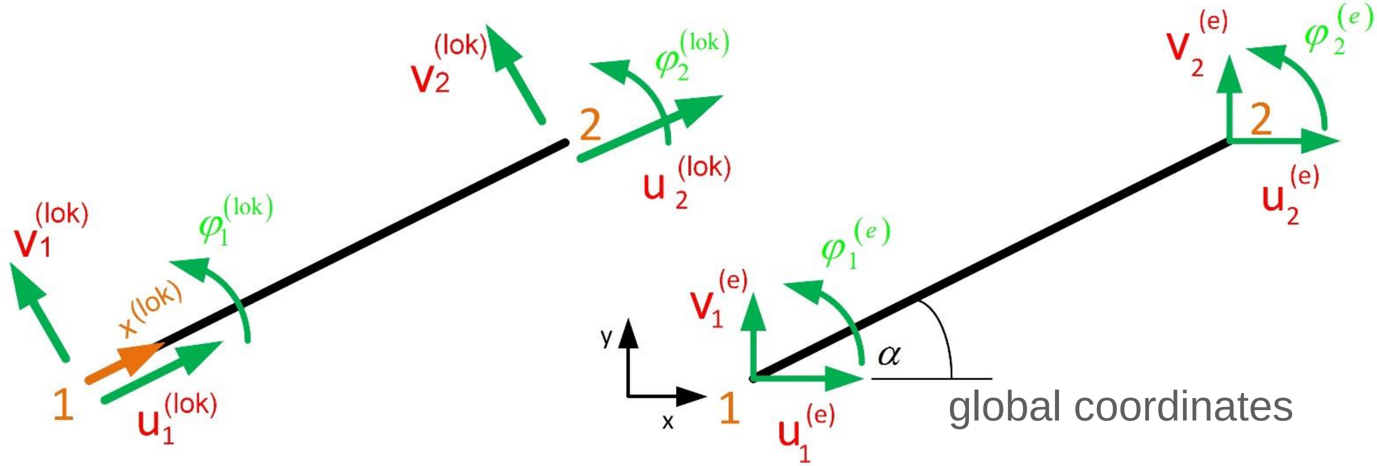
$$u^{(lok)} = u \cdot \cos\alpha + v \cdot \sin\alpha$$

$$v^{(lok)} = -u \cdot \sin\alpha + v \cdot \cos\alpha$$

Coordinate transformation

Transformation of the nodal point displacements

Local coordinates



Coordinate transformation for a beam element

$$u_1^{(lok)} = u_1^{(e)} \cdot \cos\alpha + v_1^{(e)} \cdot \sin\alpha$$

$$v_1^{(lok)} = -u_1^{(e)} \cdot \sin\alpha + v_1^{(e)} \cdot \cos\alpha$$

$$\varphi_1^{(lok)} = \varphi_1^{(e)}$$

$$u_2^{(lok)} = u_2^{(e)} \cdot \cos\alpha + v_2^{(e)} \cdot \sin\alpha$$

$$v_2^{(lok)} = -u_2^{(e)} \cdot \sin\alpha + v_2^{(e)} \cdot \cos\alpha$$

$$\varphi_2^{(lok)} = \varphi_2^{(lok)}$$

Coordinate transformation

Transformation of nodal point displacements in matrix notation

$$u_1^{(lok)} = u_1^{(e)} \cdot \cos\alpha + v_1^{(e)} \cdot \sin\alpha \qquad u_2^{(lok)} = u_2^{(e)} \cdot \cos\alpha + v_2^{(e)} \cdot \sin\alpha$$

$$v_1^{(lok)} = -u_1^{(e)} \cdot \sin\alpha + v_1^{(e)} \cdot \cos\alpha \qquad v_2^{(lok)} = -u_2^{(e)} \cdot \sin\alpha + v_2^{(e)} \cdot \cos\alpha$$

$$\varphi_1^{(lok)} = \varphi_1^{(e)} \qquad \varphi_2^{(lok)} = \varphi_2^{(e)}$$

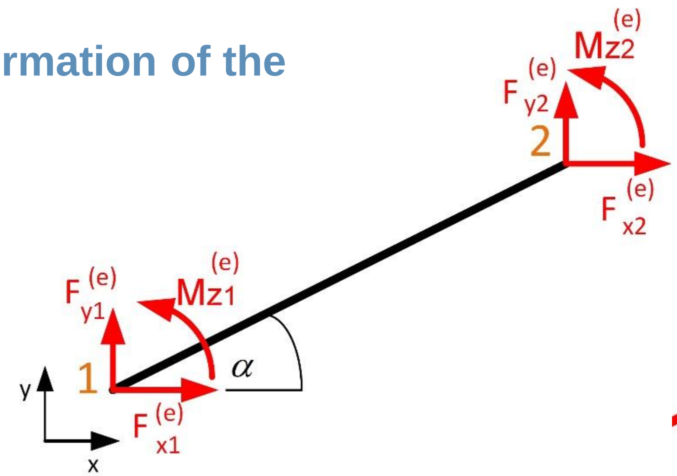
$$\begin{bmatrix} u_1^{(lok)} \\ v_1^{(lok)} \\ \varphi_1^{(lok)} \\ u_2^{(lok)} \\ v_2^{(lok)} \\ \varphi_2^{(lok)} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ \varphi_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \\ \varphi_2^{(e)} \end{bmatrix}$$

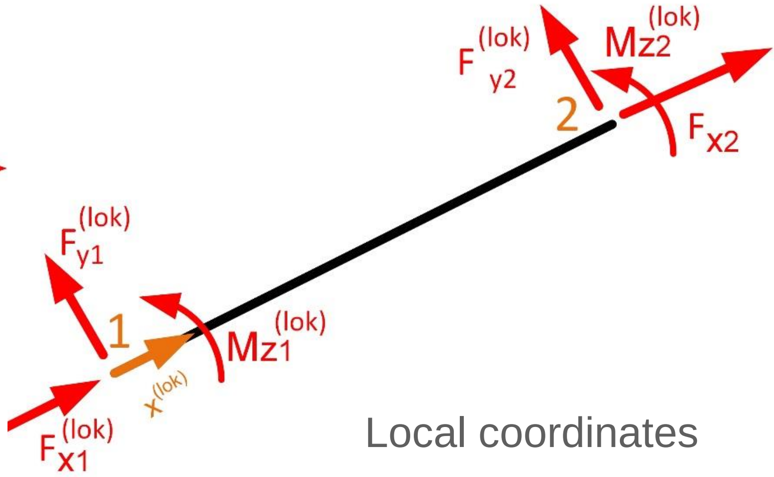
$$\underline{u}^{(lok)} = \underline{T} \cdot \underline{u}^{(e)}$$

Coordinate transformation

Coordinate transformation of the nodal forces

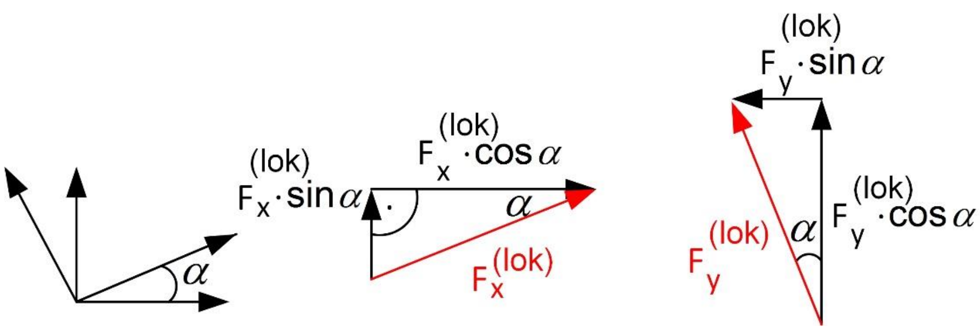
global coordinates





Local coordinates

Coordinate transformation



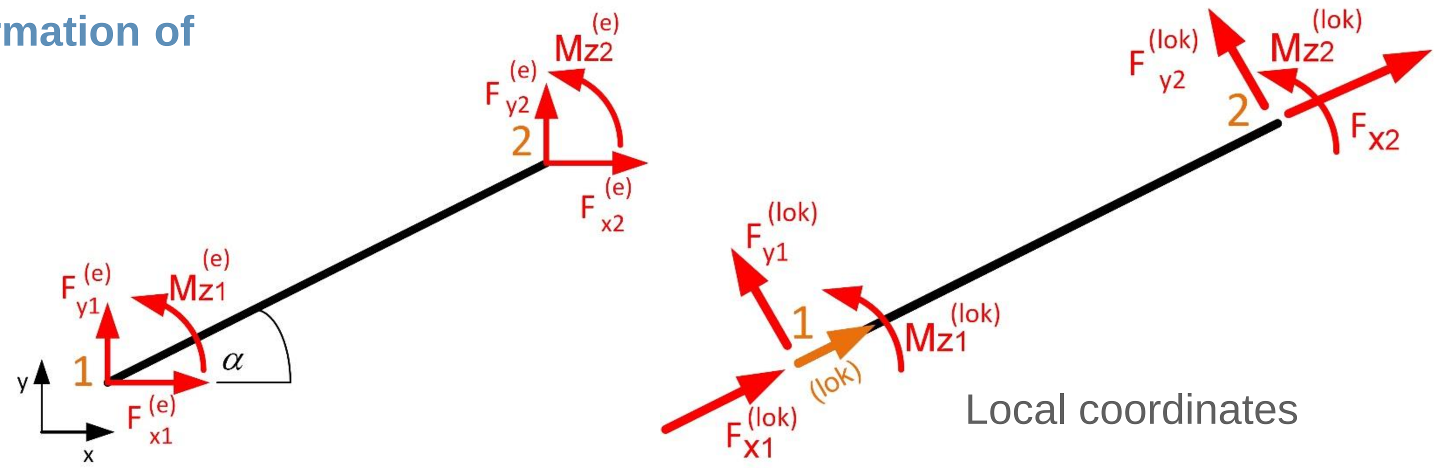
$$F_x = F_x^{(lok)} \cdot \cos\alpha - F_y^{(lok)} \cdot \sin\alpha$$

$$F_y = F_x^{(lok)} \cdot \sin\alpha + F_y^{(lok)} \cdot \cos\alpha$$

Coordinate transformation

Coordinate transformation of the nodal forces

global coordinates



Local coordinates

Coordinate transformation for a beam element

$$F_{x1}^{(e)} = \cos\alpha \cdot F_{x1}^{(lok)} - \sin\alpha \cdot F_{y1}^{(lok)}$$

$$F_{y1}^{(e)} = \sin\alpha \cdot F_{x1}^{(lok)} + \cos\alpha \cdot F_{y1}^{(lok)}$$

$$M_{z1}^{(e)} = M_{z1}^{(lok)}$$

$$F_{x2}^{(e)} = \cos\alpha \cdot F_{x2}^{(lok)} - \sin\alpha \cdot F_{y2}^{(lok)}$$

$$F_{y2}^{(e)} = \sin\alpha \cdot F_{y2}^{(lok)} + \cos\alpha \cdot F_{y2}^{(lok)}$$

$$M_{z2}^{(e)} = M_{z2}^{(lok)}$$

Coordinate transformation

Coordinate transformation of the nodal forces in matrix notation

$$F_{x1}^{(e)} = \cos\alpha \cdot F_{x1}^{(lok)} - \sin\alpha \cdot F_{y1}^{(lok)}$$

$$F_{y1}^{(e)} = \sin\alpha \cdot F_{x1}^{(lok)} + \cos\alpha \cdot F_{y1}^{(lok)}$$

$$M_{z1}^{(e)} = M_{z1}^{(lok)}$$

$$F_{x2}^{(e)} = \cos\alpha \cdot F_{x2}^{(lok)} - \sin\alpha \cdot F_{y2}^{(lok)}$$

$$F_{y2}^{(e)} = \sin\alpha \cdot F_{y2}^{(lok)} + \cos\alpha \cdot F_{y2}^{(lok)}$$

$$M_{z2}^{(e)} = M_{z2}^{(lok)}$$

$$\begin{bmatrix} F_{x1}^{(e)} \\ F_{y1}^{(e)} \\ M_{z1}^{(e)} \\ F_{x2}^{(e)} \\ F_{y2}^{(e)} \\ M_{z2}^{(e)} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} F_{x1}^{(lok)} \\ F_{y1}^{(lok)} \\ M_{z1}^{(lok)} \\ F_{x2}^{(lok)} \\ F_{y2}^{(lok)} \\ M_{z2}^{(lok)} \end{bmatrix}$$

Coordinate transformation

Displacements: $\underline{u}^{(lok)} = \underline{T} \cdot \underline{u}^{(e)}$

Stiffness matrix: $\underline{F}^{(lok)} = \underline{K}^{(lok)} \cdot \underline{u}^{(lok)}$

Forces: $\underline{F}^{(e)} = \underline{T}^{T} \cdot \underline{F}^{(lok)}$

$$\underline{F}^{(e)} = \underline{T}^{T} \cdot \underline{F}^{(lok)} = \underline{T}^{T} \cdot \underline{K}^{(lok)} \cdot \underline{u}^{(lok)} = \underline{T}^{T} \cdot \underline{K}^{(lok)} \cdot \underline{T} \cdot \underline{u}^{(e)}$$

Element stiffness matrix

$$\underline{F}^{(e)} = \underline{K}^{(e)} \cdot \underline{u}^{(e)} \quad \text{with} \quad \underline{K}^{(e)} = \underline{T}^{T} \cdot \underline{K}^{(lok)} \cdot \underline{T}$$

Element stiffness matrix in global coordinates

Element stiffness matrix in local coordinates

End

Introduction

2 Truss and beam structures

Plate and shell structures

Modeling

Stiffness and section forces matrix of a beam

Beam element