Finite Elements in Structural Analysis

Introduction

2 Truss and beam structures

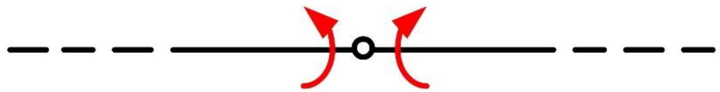
Plate and shell structures

Modeling

Hinges

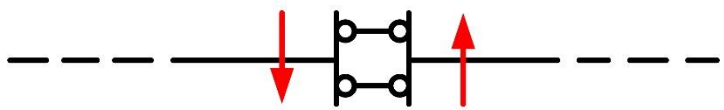
Types of hinges

Bending hinge



removed bonding: Moment

Shear force hinge

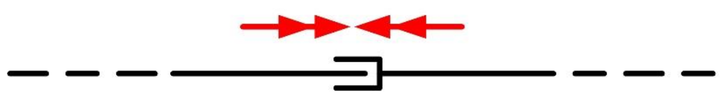


removed bonding: Shear force

Normal force hinge

removed bonding: Normal force

Torsional hinge

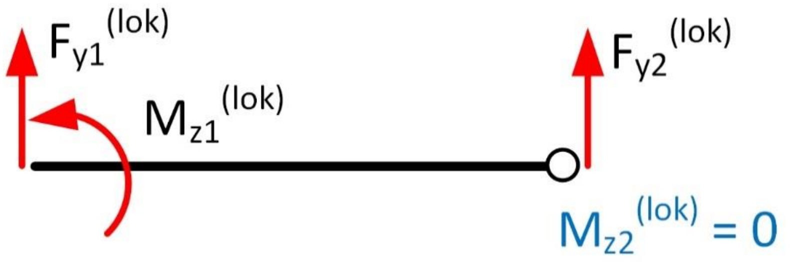


removed bonding: Torsion moment

Hinges

Example: Stiffness matrix for a bending hinge

Beam element with a bending hinge



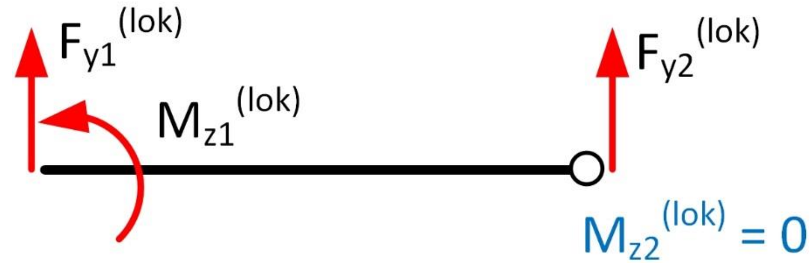
Consideration of the hinge

- The beam possesses no stiffness in the degree of freedom whose bonding (hinge) was removed.
- This requires a transformation of the stiffness matrix
- Any application of a load is not possible in the released degree of freedom

Hinges

Example: Stiffness matrix for a bending hinge

Beam element with hinge



Stiffness matrix without a hinge:

$$\frac{E\cdot I}{\ell}\cdot\begin{bmatrix} 12/\ell^2 & 6/\ell & -12/\ell^2 & 6/\ell \\ 6/\ell & 4 & -6/\ell & 2 \\ -12/\ell^2 & -6/\ell & 12/\ell^2 & -6/\ell \\ 6/\ell & 2 & -6/\ell & 4 \end{bmatrix}\cdot\begin{bmatrix} v_1^{(lok)} \\ \varphi_1^{(lok)} \\ v_2^{(lok)} \\ \varphi_2^{(lok)} \end{bmatrix} = \begin{bmatrix} F_{y1}^{(lok)} \\ M_{z1}^{(lok)} \\ F_{y2}^{(lok)} \\ M_{z2}^{(lok)} \end{bmatrix}$$

Row 4 for $M_{z2}^{(lok)}$ yields:

$$E\cdot I/\ell\cdot(6/\ell\cdot v_1^{(lok)} + 2\cdot\varphi_1^{(lok)} - 6/\ell\cdot v_2^{(lok)} + 4\cdot\varphi_2^{(lok)}) = M_{z2}^{(lok)} = 0$$

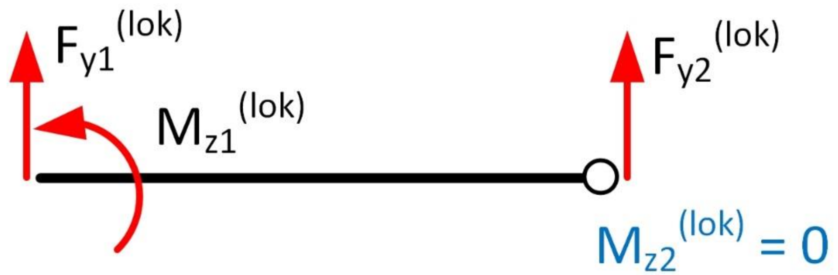
$$\rightarrow \quad \varphi_2^{(lok)} = -3/(2\cdot\ell)\cdot v_1^{(lok)} - 1/2\cdot\varphi_1^{(lok)} + 3/(2\cdot\ell)\cdot v_2^{(lok)}$$

Applying in the rows 1-3 obtain the transformed stiffness matrix

Hinges

Example: Stiffness matrix for a bending hinge

Beam element with hinge



Stiffness matrix of a beam element with a bending hinge at nodal point 2:

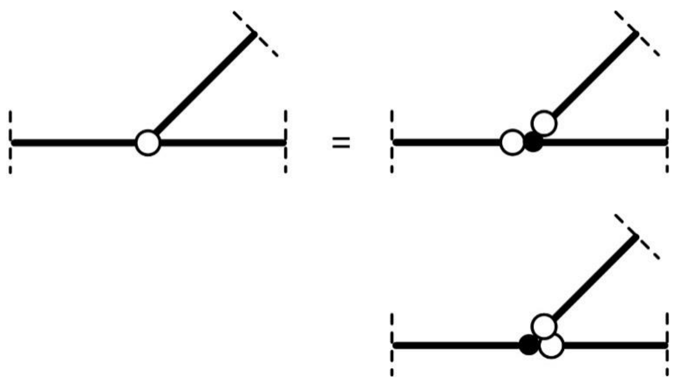
$$\frac{E\cdot I}{\ell}\begin{bmatrix} 3/\ell^2 & 3/\ell & -3/\ell^2 \\ 3/\ell & 3 & -3/\ell \\ -3/\ell^2 & -3/\ell & 3/\ell^2 \end{bmatrix}\cdot\begin{bmatrix} v_1^{(lok)} \\ \varphi_1^{(lok)} \\ v_2^{(lok)} \end{bmatrix}=\begin{bmatrix} F_{y1}^{(lok)} \\ M_{z1}^{(lok)} \\ F_{y2}^{(lok)} \end{bmatrix}$$

Hinges

Practical hints

- Faulty definition of hinges may lead to kinematic mechanisms.
- At nodes with several adjoining beam elements with moment hinges it must be ensured that the rotational degree of freedom posesses some stiffness (i.e. it is connected to an element without a hinge) or is fixed.
- If in the analysis of a beam structure with hinges the error message „kinematic mechanism" appears the hinge definitions must be checked carefully.

Example



2 Beams with a hinge
->1 Beam without a hinge

3 Beams with a hinge
-> Degree of freedom is fixed

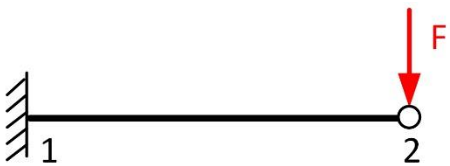
Hinges

Example: Faulty definition of a hinge at the end of a cantilever

Support Conditions

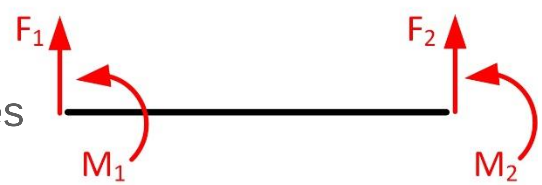
System of equations with constraints:

System



Displacement

External forces



$$\begin{bmatrix} 3/\ell^2 & 3/\ell & -3/\ell^2 & 0 \\ 3/\ell & 3 & 3/\ell & 0 \\ -3/\ell^2 & -3/\ell & 3/\ell^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix}$$

$$v_1 = 0$$

$$\varphi_1 = 0$$

$$V_1 = 0 \quad \varphi_1 = 0$$

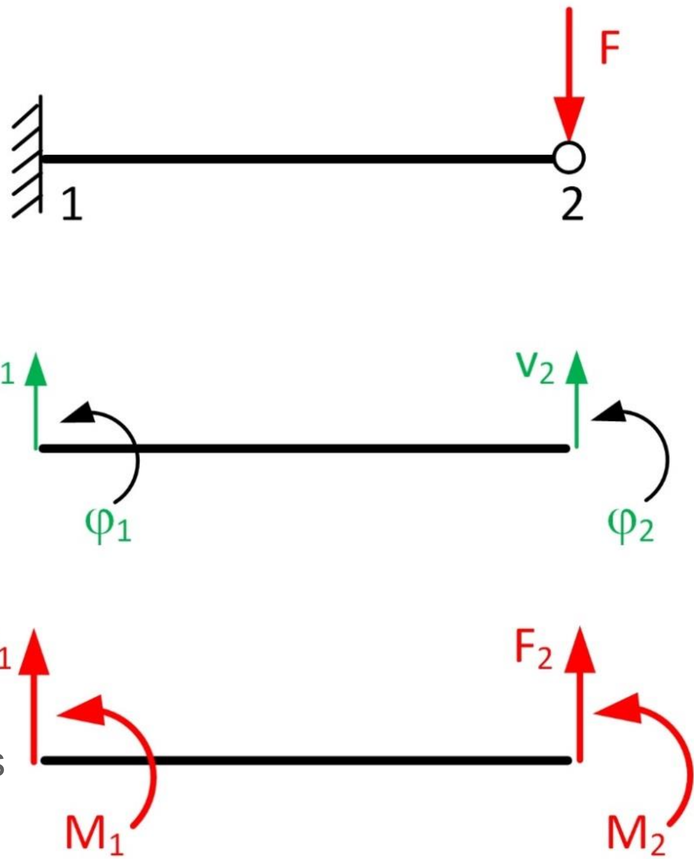
Hinges

Example: Faulty definition of a hinge at the end of a cantilever

System

Displacements

External forces



System of equations:

$$\frac{EI}{\ell}\begin{bmatrix}3/\ell^2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_2\\ \varphi_2\end{bmatrix}=\begin{bmatrix}F_2\\ M_2\end{bmatrix}$$

Properties of the system of equations:

- Singular stiffness matrix
- System of equations cannot be solved
- Application of a moment **M_2** is not meaningfull

Hinges

Example: Faulty definition of a hinge at the end of a cantilever

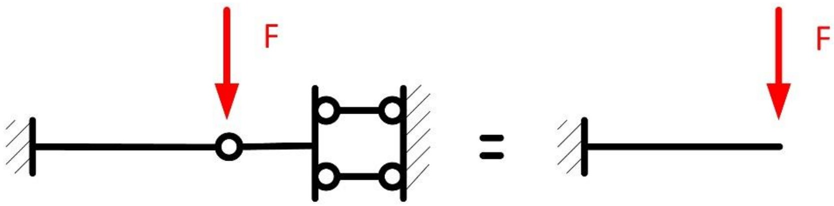
Remedies:

a) Fixing of the degree of freedom

$$\frac{EI}{\ell}\begin{bmatrix} 3/\ell^2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}=\begin{bmatrix} F_2 \\ M_2 \end{bmatrix}$$

$F_2 = -F$

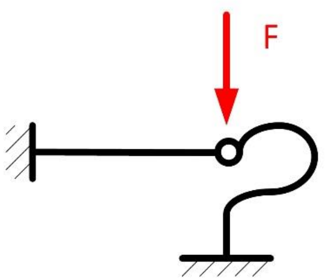
$\varphi_2 = 0$



$$3\cdot EI/\ell^3\cdot v_2=-F$$

$$v_2=-\frac{F\cdot\ell^3}{3\cdot EI}$$

b) Rigid rotational spring at point 2



$$\begin{bmatrix} 3\cdot EI/\ell^3 & 0 \\ 0 & k_\varphi \end{bmatrix}\begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix}=\begin{bmatrix} -F \\ M_2 \end{bmatrix}$$

Regular stiffness matrix

Example: Faulty definition of a hinge at the end of a cantilever

Remedies:

c) Elimination of the (faulty) hinge definition

=

$$\frac{E \cdot I}{\ell} \cdot \begin{bmatrix} 12/\ell^2 & 6/\ell & -12/\ell^2 & 6/\ell \\ 6/\ell & 4 & -6/\ell & 2 \\ -12/\ell^2 & -6/\ell & 12/\ell^2 & -6/\ell \\ 6/\ell & 2 & -6/\ell & 4 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \end{bmatrix}$$

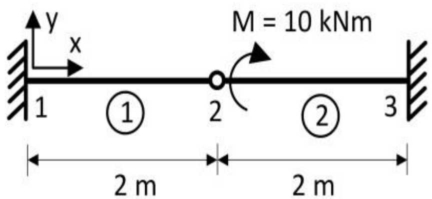
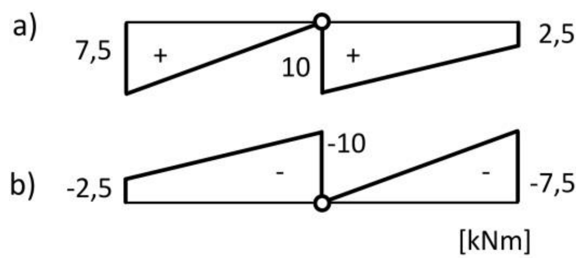
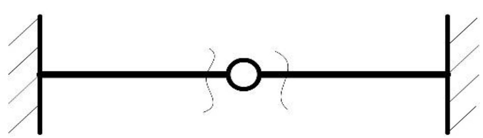
Global stiffness matrix

$$\frac{EI}{\ell} \cdot \begin{bmatrix} 12/\ell^2 & -6/\ell \\ -6/\ell & 4 \end{bmatrix} \cdot \begin{bmatrix} v_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} F_{y2} \\ M_{z2} \end{bmatrix}$$

- Stiffness matrix is regular

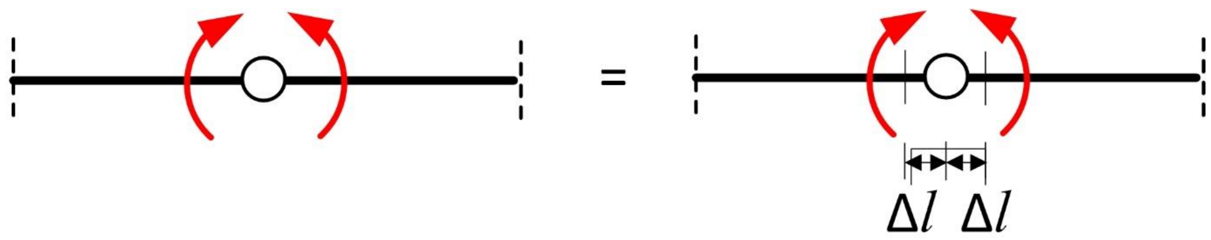
Hinges

Example: Two beam elements with a bending hinge

Hinge definition		Consequence
System		Moment M is introduced as nodal load for nodal point 2
Hinge in beam 1		M applied on beam element 2
Hinge in beam 2		M applied on beam element 1
Hinge in beam 1 + 2		No moment diagram! Forbidden kinematic mechanism for φ_2

Hinges

Example: Moment pair acting at a bending hinge



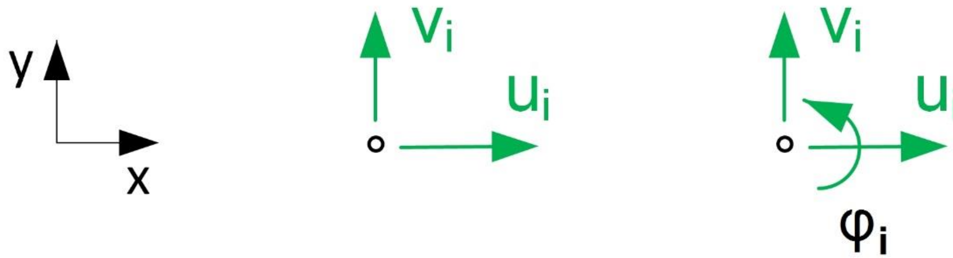
Problem: At the nodal point the moment pair cannot be applied.

Solution:

- Insert two short beams e.g. $\Delta l = 1cm$
- Define moments as element loads at both beam elements.

Degrees of freedom of plane truss and beam systems

2D systems



2 degrees of freedom for each nodal point: u_i and v_i

Nodal points which are connected only with truss elements and displacement springs.

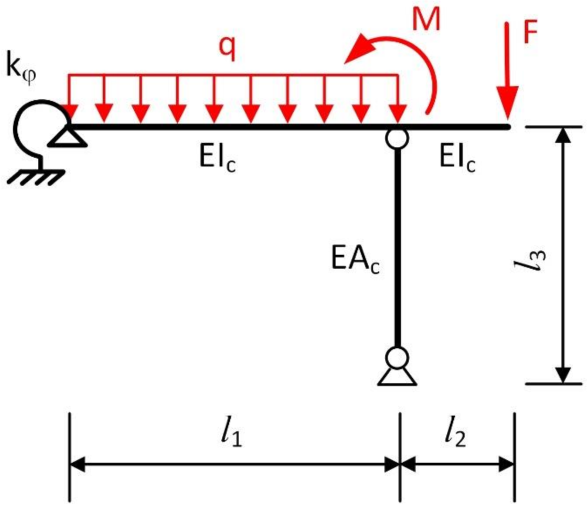
3 Degrees of freedom for each nodal point: u_i, v_i and φ_i

Nodal points which are connected with beam elements or rotation springs.

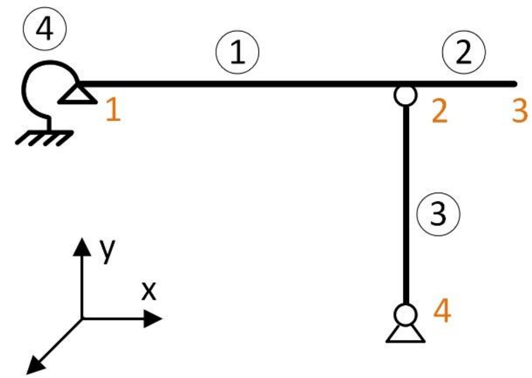
All other degrees of freedom are to be fixed in order to avoid a singular global stiffness matrix.

Example: Structural system

System



Nodal points and elements



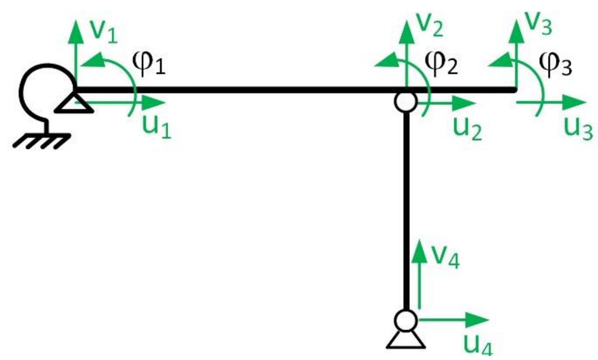
Element types

Elements 1,2 : Beam elements

Element 3 : Truss element

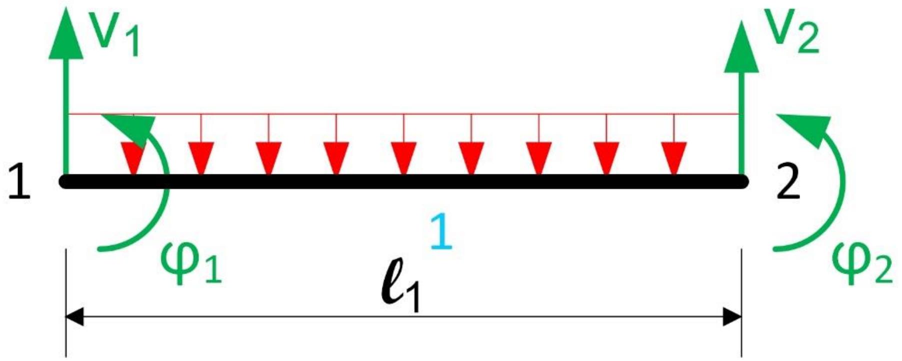
Element 4 : Spring element

Deegrees of freedom



Example: Structural system

Element 1: Stiffness matrix of a beam element

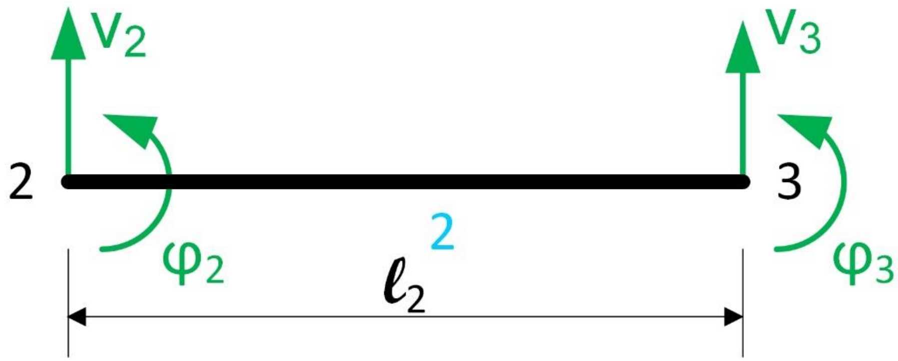


$$\frac{E \cdot I_c}{\ell_1} \cdot \begin{bmatrix} 4 & -\frac{6}{\ell_1} & 2 \\ -\frac{6}{\ell_1} & \frac{12}{\ell_1^2} & -\frac{6}{\ell_1} \\ 2 & -\frac{6}{\ell_1} & 4 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 \\ v_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} M_{z1}^{(1)} \\ F_{y2}^{(1)} \\ M_{z2}^{(1)} \end{bmatrix} - \begin{bmatrix} \frac{q \cdot \ell_1^2}{12} \\ \frac{q \cdot \ell_1}{2} \\ -\frac{q \cdot \ell_1^2}{12} \end{bmatrix}$$

Restraint condition **v_1=0** already considered in the element stiffness matrix

Example: Structural system

Element 2: Stiffness matrix of a beam element



$$\frac{E\cdot I_c}{\ell_2}\cdot\begin{bmatrix} \frac{12}{\ell_2^{\,2}} & \frac{6}{\ell_2} & -\frac{12}{\ell_2^{\,2}} & \frac{6}{\ell_2} \\ \frac{6}{\ell_2} & 4 & -\frac{6}{\ell_2} & 2 \\ -\frac{12}{\ell_2^{\,2}} & -\frac{6}{\ell_2} & \frac{12}{\ell_2^{\,2}} & -\frac{6}{\ell_2} \\ \frac{6}{\ell_2} & 2 & -\frac{6}{\ell_2} & 4 \end{bmatrix}\cdot\begin{bmatrix} v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3 \end{bmatrix}=\begin{bmatrix} F_{y2}^{(2)} \\ M_{z2}^{(2)} \\ F_{y3}^{(2)} \\ M_{z3}^{(2)} \end{bmatrix}$$

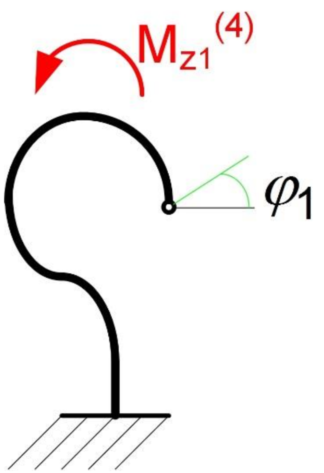
Example: Structural system

Element 3: Stiffness matrix of a truss element

$$\frac{E \cdot A_c}{\ell_3} \cdot v_2 = F_{y2}^{(3)}$$

Element 4: Stiffness of a rotational spring

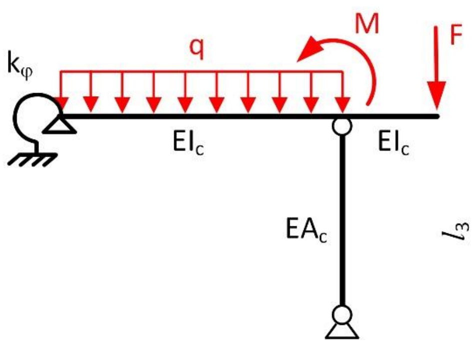
$$M_{z1}^{(4)} = k_\varphi \cdot \varphi_1$$

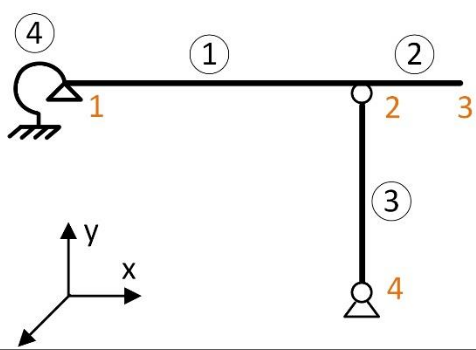


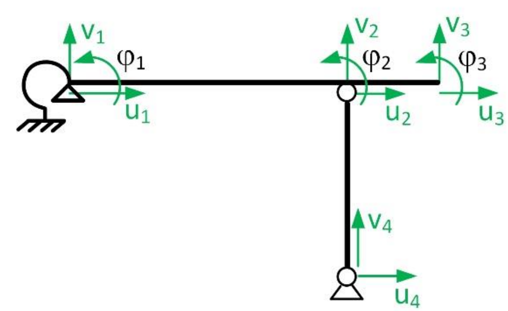
Example: Structural system

Matrix with restraints: $u_1 = 0 \quad u_2 = 0 \quad u_3 = 0 \quad v_1 = 0 \quad u_4 = 0 \quad v_4 = 0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 \\ v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ M \\ -F \\ 0 \end{bmatrix}$$







Example: Structural system

Addition of element 1

$$\begin{bmatrix} 4\cdot c_1 & -6\cdot\frac{c_1}{l_1} & 2\cdot c_1 & 0 & 0 \\ -6\cdot\frac{c_1}{l_1} & 12\cdot\frac{c_1}{l_1^{\,2}} & -6\cdot\frac{c_1}{l_1} & 0 & 0 \\ 2\cdot c_1 & -6\cdot\frac{c_1}{l_1} & 4\cdot c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 \\ v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} -q\cdot\frac{l_1^{\,2}}{12} \\ -q\cdot\frac{l_1}{2} \\ M+q\cdot\frac{l_1^{\,2}}{12} \\ -F \\ 0 \end{bmatrix}$$

with $c_1 = EI_c/\ell_1$ and $c_2 = EI_c/\ell_2$

Element 1

Example: Structural system

Addition of element 2

$$\begin{bmatrix} 4\cdot c_1 & -6\cdot\frac{c_1}{l_1} & 2\cdot c_1 & 0 & 0 \\ -6\cdot\frac{c_1}{l_1} & 12\cdot\frac{c_1}{l_1^{\,2}}+12\cdot\frac{c_2}{l_2^{\,2}} & -6\cdot\frac{c_1}{l_1}+6\cdot\frac{c_2}{l_2} & -12\cdot\frac{c_2}{l_2^{\,2}} & 6\cdot\frac{c_2}{l_2} \\ 2\cdot c_1 & -6\cdot\frac{c_1}{l_1}+6\cdot\frac{c_2}{l_2} & 4\cdot c_1 \text{ -}4\cdot c_2 & -6\cdot\frac{c_2}{l_2} & 2\cdot c_2 \\ 0 & -12\cdot\frac{c_2}{l_2^{\,2}} & -6\cdot\frac{c_2}{l_2} & 12\cdot\frac{c_2}{l_2^{\,2}} & -6\cdot\frac{c_2}{l_2} \\ 0 & 6\cdot\frac{c_2}{l_2} & 2\cdot c_2 & -6\cdot\frac{c_2}{l_2} & 4\cdot c_2 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 \\ v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} -q\cdot\frac{l_1^{\,2}}{12} \\ -q\cdot\frac{l_1}{2} \\ M+q\cdot\frac{l_1^{\,2}}{12} \\ -F \\ 0 \end{bmatrix}$$

with $c_1 = EI_c / \ell_1$ and $c_2 = EI_c / \ell_2$

Element 2

Example: Structural system

Addition of element 3

$$\begin{bmatrix} +4\cdot c_1 & -6\cdot\frac{c_1}{l_1} & 2\cdot c_1 & 0 & 0 \\ -6\cdot\frac{c_1}{l_1} & 12\cdot\frac{c_1}{l_1^2}+12\cdot\frac{c_2}{l_2^2}+\frac{EA_c}{l_3} & -6\cdot\frac{c_1}{l_1}+6\cdot\frac{c_2}{l_2} & -12\cdot\frac{c_2}{l_2^2} & 6\cdot\frac{c_2}{l_2} \\ 2\cdot c_1 & -6\cdot\frac{c_1}{l_1}+6\cdot\frac{c_2}{l_2} & 4\cdot c_1+4\cdot c_2 & -6\cdot\frac{c_2}{l_2} & 2\cdot c_2 \\ 0 & -12\cdot\frac{c_2}{l_2^2} & -6\cdot\frac{c_2}{l_2} & 12\cdot\frac{c_2}{l_2^2} & -6\cdot\frac{c_2}{l_2} \\ 0 & 6\cdot\frac{c_2}{l_2} & 2\cdot c_2 & -6\cdot\frac{c_2}{l_2} & 4\cdot c_2 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 \\ v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} -q\cdot\frac{l_1^2}{12} \\ -q\cdot\frac{l_1}{2} \\ M+q\cdot\frac{l_1^2}{12} \\ -F \\ 0 \end{bmatrix}$$

with $c_1 = EI_c / \ell_1$ and $c_2 = EI_c / \ell_2$

Element 3

Example: Structural system

Addition of element 4

$$\begin{bmatrix} k_\varphi + 4\cdot c_1 & -6\cdot\frac{c_1}{l_1} & 2\cdot c_1 & 0 & 0 \\ -6\cdot\frac{c_1}{l_1} & 12\cdot\frac{c_1}{l_1^2}+12\cdot\frac{c_2}{l_2^2}+\frac{EA_c}{l_3} & -6\cdot\frac{c_1}{l_1}+6\cdot\frac{c_2}{l_2} & -12\cdot\frac{c_2}{l_2^2} & 6\cdot\frac{c_2}{l_2} \\ 2\cdot c_1 & -6\cdot\frac{c_1}{l_1}+6\cdot\frac{c_2}{l_2} & 4\cdot c_1+4\cdot c_2 & -6\cdot\frac{c_2}{l_2} & 2\cdot c_2 \\ 0 & -12\cdot\frac{c_2}{l_2^2} & -6\cdot\frac{c_2}{l_2} & 12\cdot\frac{c_2}{l_2^2} & -6\cdot\frac{c_2}{l_2} \\ 0 & 6\cdot\frac{c_2}{l_2} & 2\cdot c_2 & -6\cdot\frac{c_2}{l_2} & 4\cdot c_2 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 \\ v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} -q\cdot\frac{l_1^2}{12} \\ -q\cdot\frac{l_1}{2} \\ M+q\cdot\frac{l_1^2}{12} \\ -F \\ 0 \end{bmatrix}$$

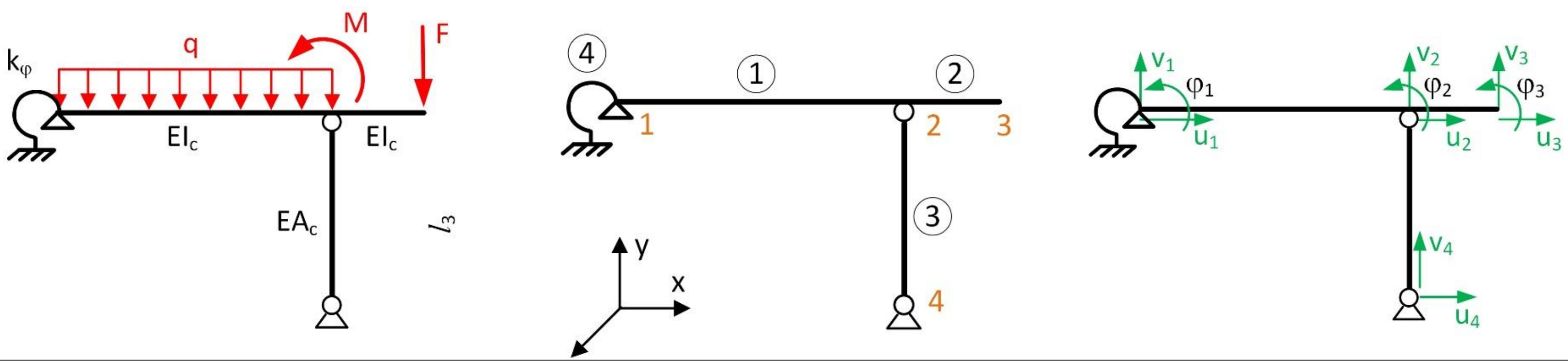
with $c_1 = EI_c / \ell_1$ and $c_2 = EI_c / \ell_2$

Element 4

Example: Structural system

Global stiffness matrix

$$
\begin{bmatrix}
k_\varphi + 4\cdot c_1 & -6\cdot\frac{c_1}{l_1} & 2\cdot c_1 & 0 & 0 \\
-6\cdot\frac{c_1}{l_1} & 12\cdot\frac{c_1}{l_1^2}+12\cdot\frac{c_2}{l_2^2}+\frac{EA_c}{l_3} & -6\cdot\frac{c_1}{l_1}+6\cdot\frac{c_2}{l_2} & -12\cdot\frac{c_2}{l_2^2} & 6\cdot\frac{c_2}{l_2} \\
2\cdot c_1 & -6\cdot\frac{c_1}{l_1}+6\cdot\frac{c_2}{l_2} & 4\cdot c_1+4\cdot c_2 & -6\cdot\frac{c_2}{l_2} & 2\cdot c_2 \\
0 & -12\cdot\frac{c_2}{l_2^2} & -6\cdot\frac{c_2}{l_2} & 12\cdot\frac{c_2}{l_2^2} & -6\cdot\frac{c_2}{l_2} \\
0 & 6\cdot\frac{c_2}{l_2} & 2\cdot c_2 & -6\cdot\frac{c_2}{l_2} & 4\cdot c_2
\end{bmatrix}
\cdot
\begin{bmatrix}
\varphi_1 \\ v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3
\end{bmatrix}
=
\begin{bmatrix}
-q\cdot\frac{l_1^2}{12} \\ -q\cdot\frac{l_1}{2} \\ M+q\cdot\frac{l_1^2}{12} \\ -F \\ 0
\end{bmatrix}
$$



3D Truss element

Element stiffness matrix

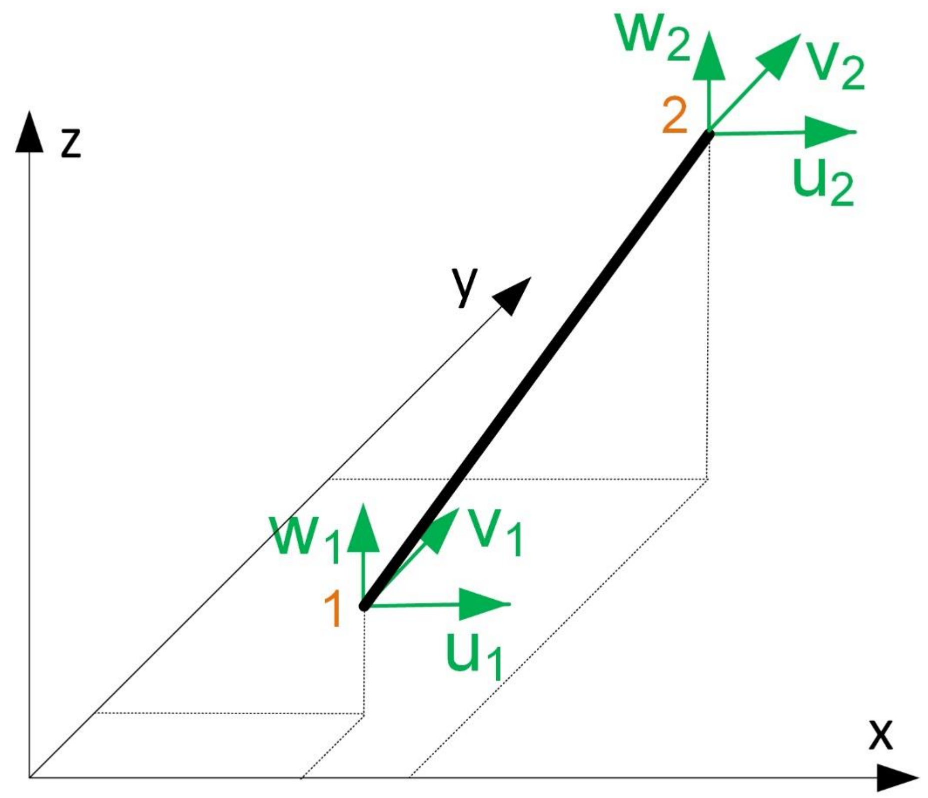
3 degrees of freedom at each nodal point:

→ 6 x 6 Matrix

Derivation:

- 3D coordinate transformation of the local stiffness matrix

$$\underline{K}_{3D} = \underline{T}_{3D}^{T} \cdot \underline{K}^{(lok)} \cdot \underline{T}_{3D}$$



3D Beam element

Element stiffness matrix

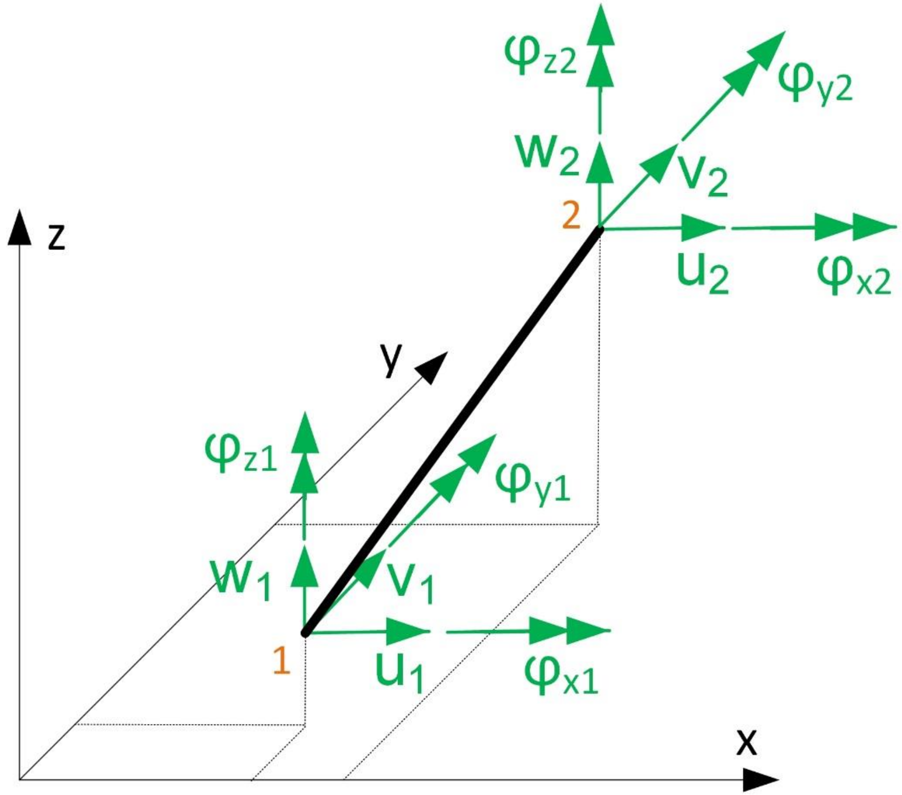
6 degrees of freedom at each nodal point

→ 12 x 12 Matrix

Derivation

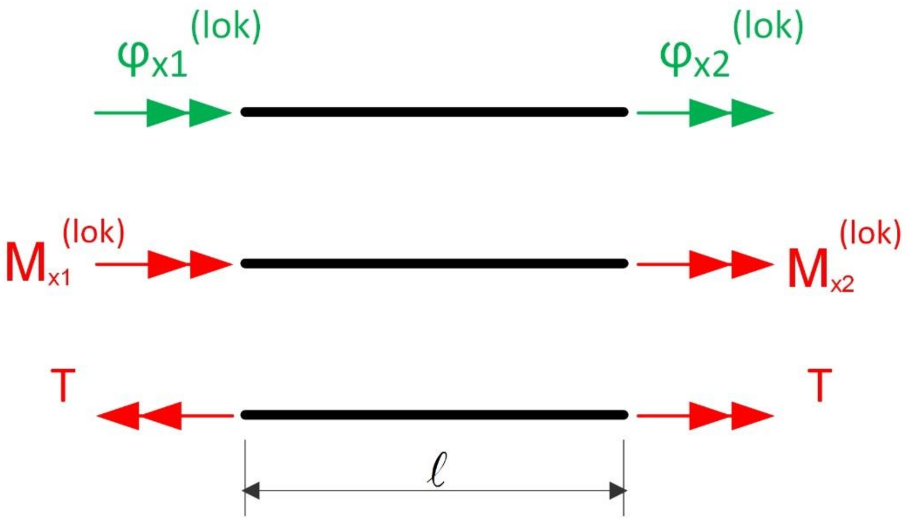
- Augmentation of the two-dimensional local stiffness matrix by transverse bending and torsion.
- 3D coordinate transformation

$$\underline{K}_{3D} = \underline{T}_{3D}^{T} \cdot \underline{K}^{(lok)} \cdot \underline{T}_{3D}$$



3D Beam element

Stiffness matrix of an element subjected to torsion (St. Venant)



$$\varphi = \left(\varphi_{x2}^{(lok)} - \varphi_{x1}^{(lok)}\right) = \frac{\ell}{G \cdot I_T} \cdot T$$

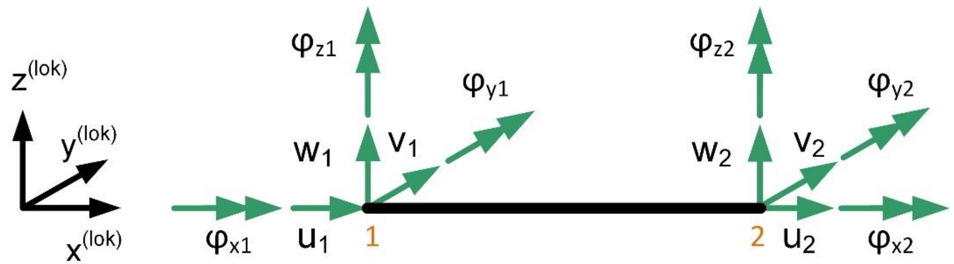
$$M_{x1}^{(lok)} = -T = \frac{G \cdot I_T}{\ell} \cdot \left(\varphi_{x1}^{(lok)} - \varphi_{x2}^{(lok)}\right)$$

$$M_{x2}^{(lok)} = T = \frac{G \cdot I_T}{\ell} \cdot \left(-\varphi_{x1}^{(lok)} + \varphi_{x2}^{(lok)}\right)$$

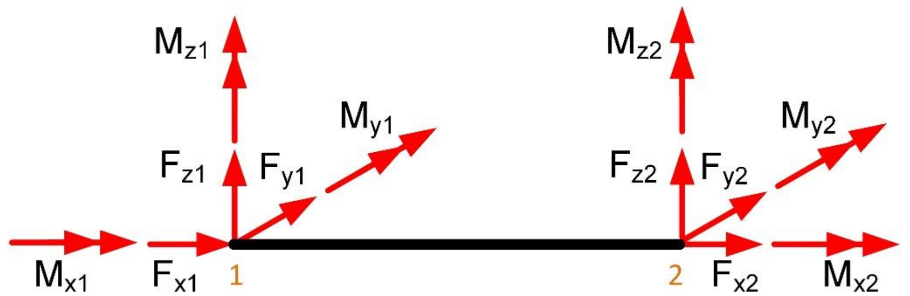
$$\frac{G \cdot I_T}{\ell} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \varphi_{x1}^{(lok)} \\ \varphi_{x2}^{(lok)} \end{bmatrix} = \begin{bmatrix} M_{x1}^{(lok)} \\ M_{x2}^{(lok)} \end{bmatrix}$$

3D Beam element

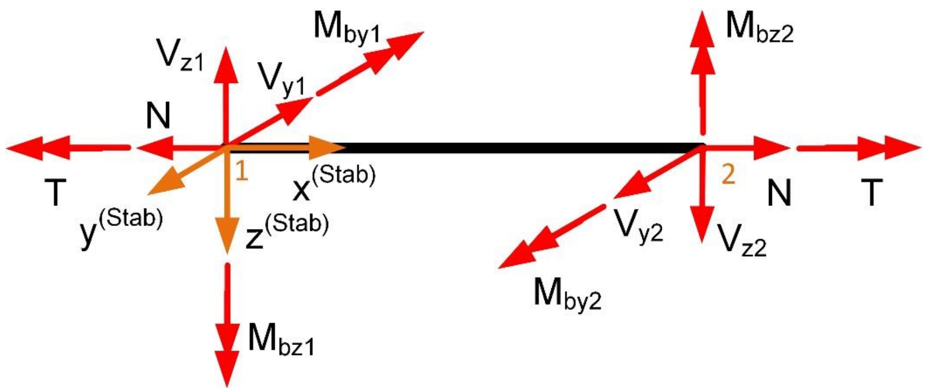
Displacements and forces



Displacements



Forces



Section forces

3D Beam element

$$
\begin{bmatrix}
\frac{E \cdot A}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{E \cdot A}{l} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12 \cdot E \cdot I_z}{l^3} & 0 & 0 & 0 & \frac{6 \cdot E \cdot I_z}{l^2} & 0 & -\frac{12 \cdot E \cdot I_z}{l^3} & 0 & 0 & 0 & \frac{6 \cdot E \cdot I_z}{l^2} \\
0 & 0 & \frac{12 \cdot E \cdot I_y}{l^3} & 0 & \frac{6 \cdot E \cdot I_z}{l^2} & 0 & 0 & 0 & -\frac{12 \cdot E \cdot I_y}{l^3} & 0 & -\frac{6 \cdot E \cdot I_z}{l^2} & 0 \\
0 & 0 & 0 & \frac{G \cdot I_T}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{G \cdot I_T}{l} & 0 & 0 \\
0 & 0 & -\frac{6 \cdot E \cdot I_y}{l^2} & 0 & \frac{4 \cdot E \cdot I_y}{l} & 0 & 0 & 0 & \frac{6 \cdot E \cdot I_y}{l^2} & 0 & \frac{2 \cdot E \cdot I_y}{l} & 0 \\
0 & \frac{6 \cdot E \cdot I_z}{l^2} & 0 & 0 & 0 & \frac{4 \cdot E \cdot I_z}{l} & 0 & -\frac{6 \cdot E \cdot I_z}{l^2} & 0 & 0 & 0 & \frac{2 \cdot E \cdot I_z}{l} \\
-\frac{E \cdot A}{l} & 0 & 0 & 0 & 0 & 0 & \frac{E \cdot A}{l} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12 \cdot E \cdot I_z}{l^3} & 0 & 0 & 0 & -\frac{6 \cdot E \cdot I_z}{l^3} & 0 & \frac{12 \cdot E \cdot I_z}{l^3} & 0 & 0 & 0 & -\frac{6 \cdot E \cdot I_z}{l^3} \\
0 & 0 & -\frac{12 \cdot E \cdot I_y}{l^3} & 0 & \frac{6 \cdot E \cdot I_y}{l^2} & 0 & 0 & 0 & \frac{12 \cdot E \cdot I_y}{l^3} & 0 & \frac{6 \cdot E \cdot I_y}{l^2} & 0 \\
0 & 0 & 0 & -\frac{G \cdot I_T}{l} & 0 & 0 & 0 & 0 & 0 & \frac{G \cdot I_T}{l} & 0 & 0 \\
0 & 0 & -\frac{6 \cdot E \cdot I_y}{l^2} & 0 & \frac{2 \cdot E \cdot I_y}{l} & 0 & 0 & 0 & \frac{6 \cdot E \cdot I_y}{l^2} & 0 & \frac{4 \cdot E \cdot I_y}{l} & 0 \\
0 & \frac{6 \cdot E \cdot I_z}{l^2} & 0 & 0 & 0 & \frac{2 \cdot E \cdot I_z}{l} & 0 & -\frac{6 \cdot E \cdot I_z}{l^2} & 0 & 0 & 0 & \frac{4 \cdot E \cdot I_z}{l}
\end{bmatrix}
\cdot
\begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \varphi_{x1} \\ \varphi_{y1} \\ \varphi_{z1} \\ u_2 \\ v_2 \\ w_2 \\ \varphi_{x2} \\ \varphi_{y2} \\ \varphi_{z2} \end{bmatrix}
=
\begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{z1} \\ M_{x1} \\ M_{y1} \\ M_{z1} \\ F_{x2} \\ F_{y2} \\ F_{z2} \\ M_{x2} \\ M_{y2} \\ M_{z2} \end{bmatrix}
$$

End

Introduction

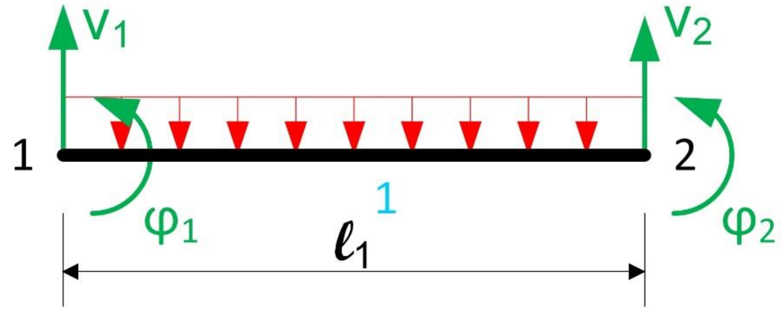
2 Truss and beam structures

Plate and shell structures

Modeling

Example: Structural system

Element 1: Stiffness matrix of a beam element

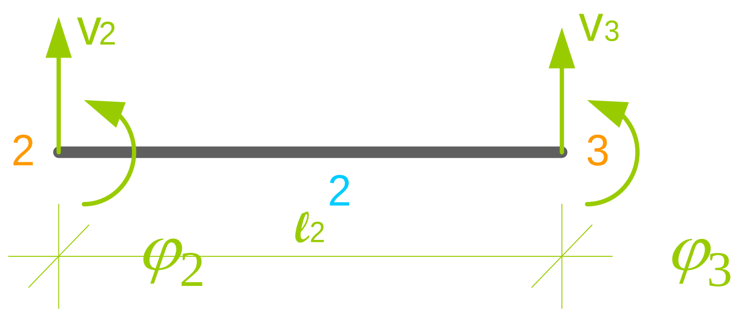


$$\frac{E \cdot I_c}{\ell_1} \cdot \begin{bmatrix} 4 & -\frac{6}{\ell_1} & 2 \\ -\frac{6}{\ell_1} & \frac{12}{\ell_1^2} & -\frac{6}{\ell_1} \\ 2 & -\frac{6}{\ell_1} & 4 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 \\ v_2 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} M_{z1}^{(1)} \\ F_{y2}^{(1)} \\ M_{z2}^{(1)} \end{bmatrix} - \begin{bmatrix} \frac{q \cdot \ell_1^2}{12} \\ \frac{q \cdot \ell_1}{2} \\ -\frac{q \cdot \ell_1^2}{12} \end{bmatrix}$$

Restraint condition **v_1=0** already considered in the element stiffness matrix

Example: Structural system

Element 2: Stiffness matrix of a beam element

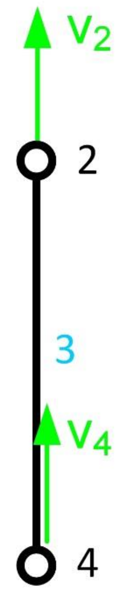


$$\frac{E \cdot I_c}{\ell_2} \cdot \begin{bmatrix} \frac{12}{\ell_2^2} & \frac{6}{\ell_2} & -\frac{12}{\ell_2^2} & \frac{6}{\ell_2} \\ \frac{6}{\ell_2} & 4 & -\frac{6}{\ell_2} & 2 \\ -\frac{12}{\ell_2^2} & -\frac{6}{\ell_2} & \frac{12}{\ell_2^2} & -\frac{6}{\ell_2} \\ \frac{6}{\ell_2} & 2 & -\frac{6}{\ell_2} & 4 \end{bmatrix} \cdot \begin{bmatrix} v_2 \\ \varphi_2 \\ v_3 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} F_{y2}^{(2)} \\ M_{z2}^{(2)} \\ F_{y3}^{(2)} \\ M_{z3}^{(2)} \end{bmatrix}$$

Example: Structural system

Element 3: Stiffness matrix of a truss element

$$\frac{E \cdot A_c}{\ell_3} \cdot v_2 = F_{y2}^{(3)}$$



Element 4: Stiffness of a rotational spring

$$M_{z1}^{(4)} = k_\varphi \cdot \varphi_1$$