Finite Elements in Structural Analysis

Introduction

2 Truss and beam structures

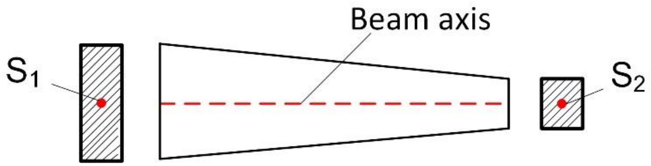
Plate and shell structures

Modeling

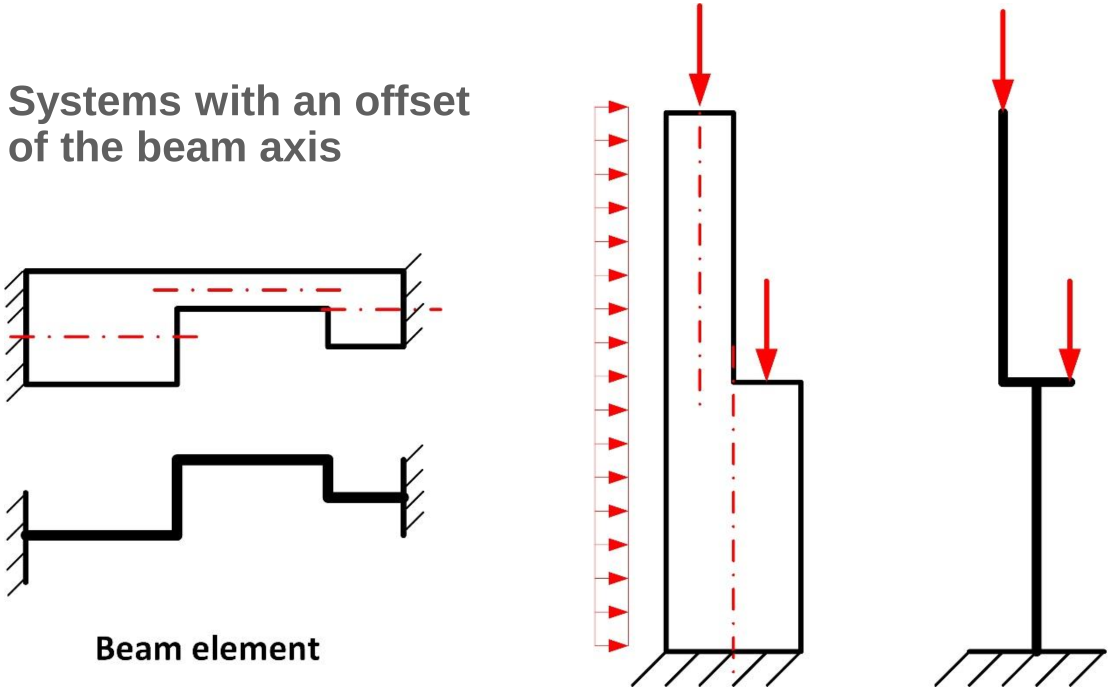
Beams in bending

Location of the beam axis

Definition: The beam axis is the line connecting the centers of gravity of the cross-sections at both faces.



Systems with an offset of the beam axis



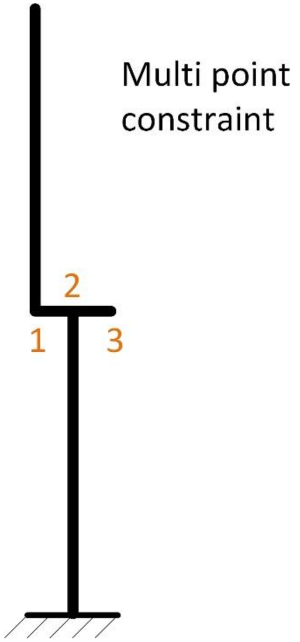
Modeling

- Multi point constraint (MPC)
- Beam elements with large stiffness
- Beam element with eccentric axis

Beams in bending

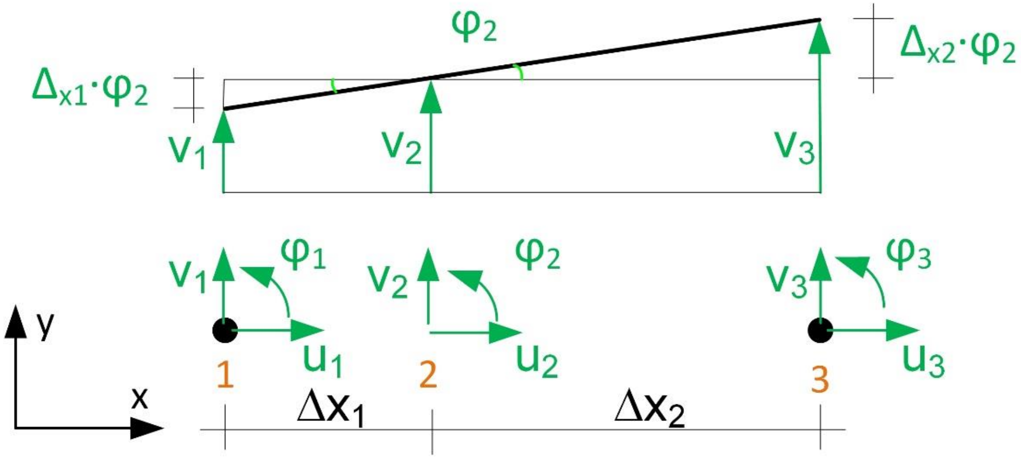
Position of the beam axis

Modeling of the eccentricity with kinematic multi-point constraints



Example:

Coupling of the nodal points 1 and 3 (Slaves) to the nodel point 2 (Master)



Coupling conditions

$$u_1 = u_2 \qquad u_3 = u_2$$

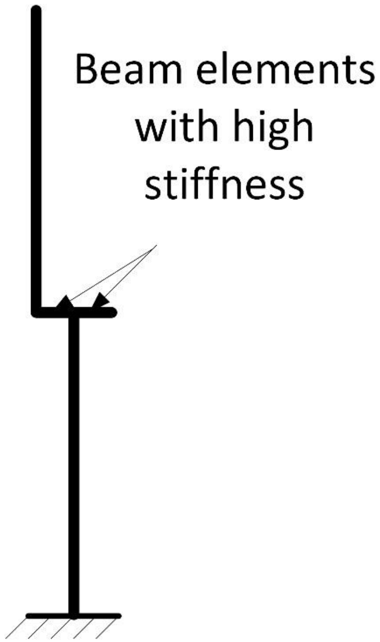
$$v_1 = v_2 - \Delta x_1 \cdot \varphi_2 \qquad v_3 = v_2 + \Delta x_2 \cdot \varphi_2$$

$$\varphi_1 = \varphi_2 \qquad \varphi_3 = \varphi_2$$

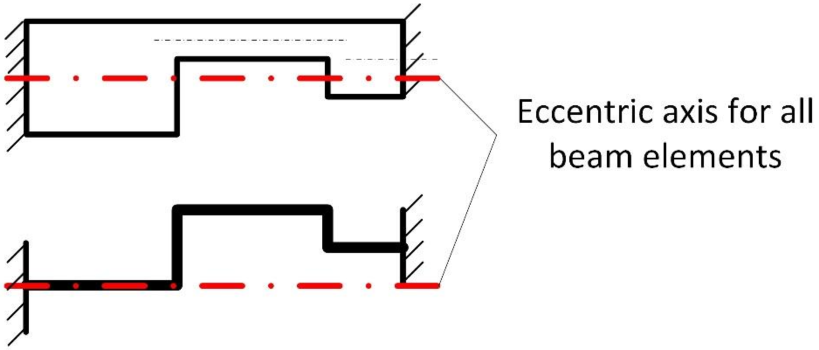
Beams in bending

Modeling of the eccentricity by beam elements with large stiffness

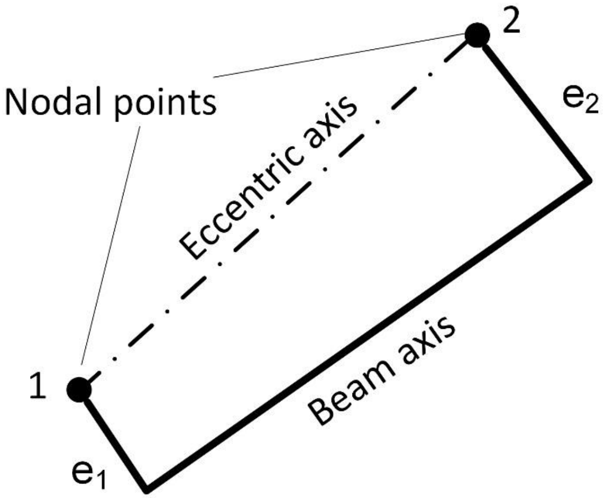
In order to avoid numerical problems, it should be ensured that the chosen stiffness of the beam elements do not differ by many orders of magnitude.



Modeling by beam elements with eccentric axis



Beam element with a jump in the beam axis

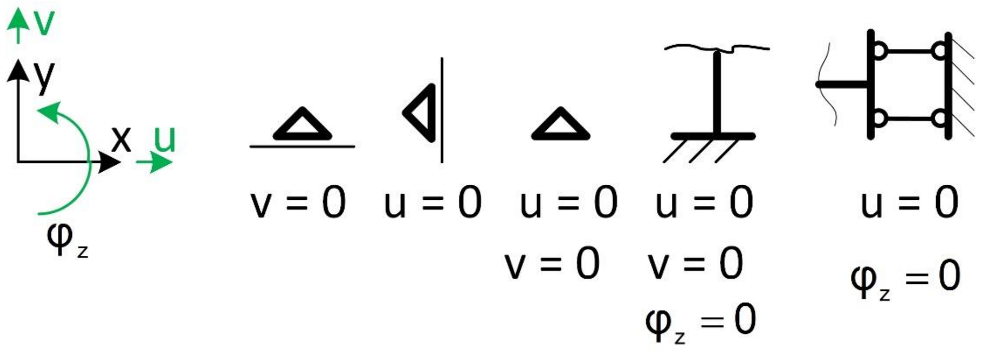


Modeling of supports

Supports in the direction of the global coordinate system

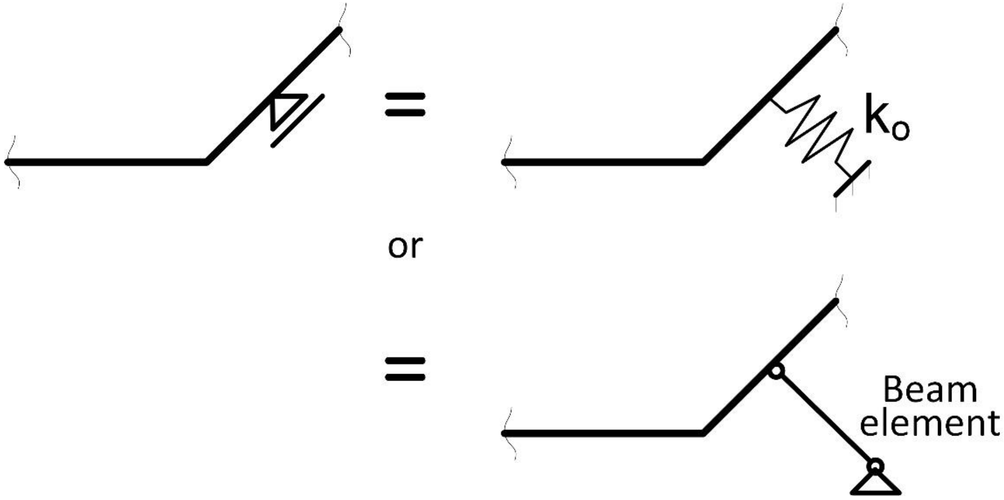
Definition by fixing individual

degrees of freedom



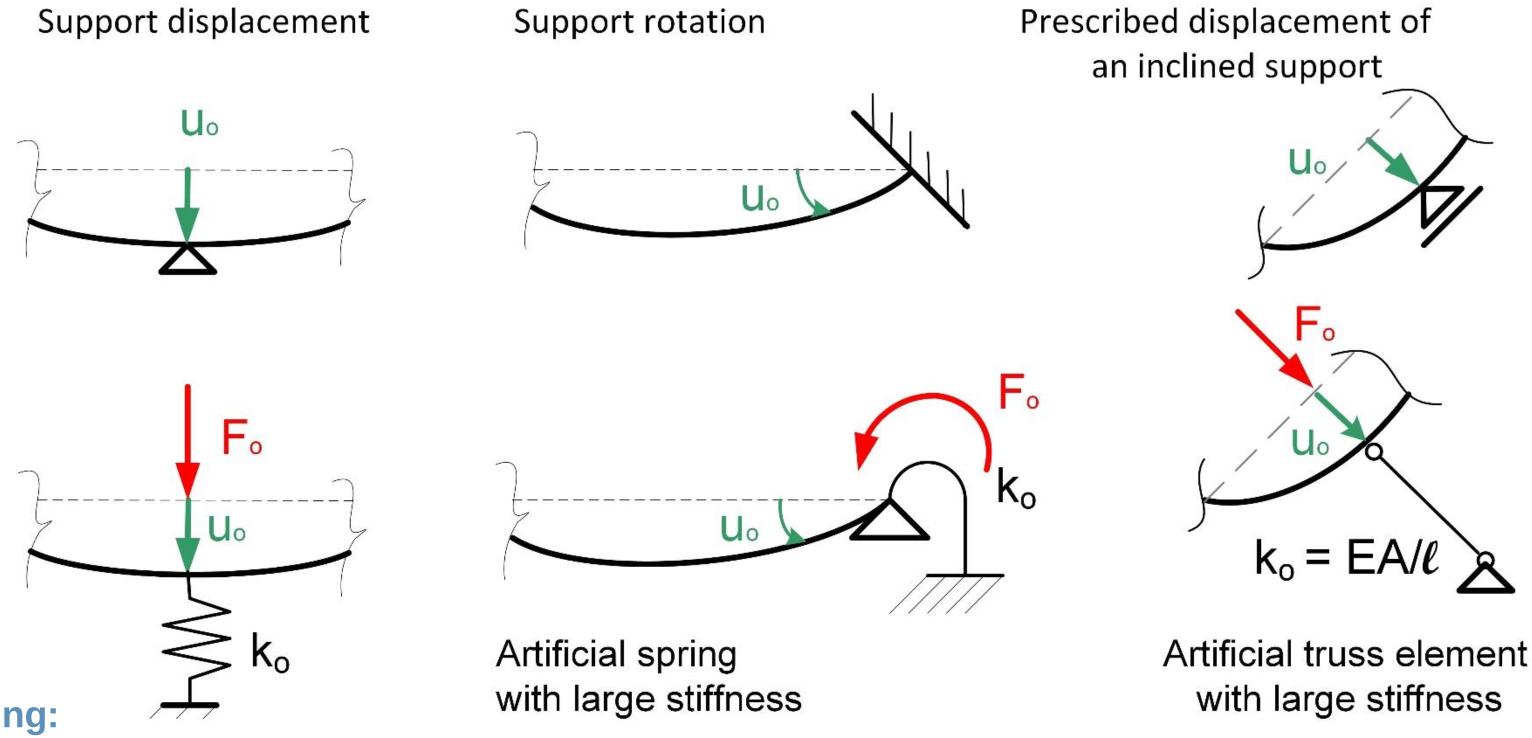
Modeling of inclined supports

- Kinematic coupling conditions
- Spring with large stiffness
- Truss element with large stiffness



Modeling of supports

Prescribed support displacements and rotations



Modeling:

- Kinematic conditions are introduced in the system of equations of the global system
- Artificial spring with large stiffness k_0 and large force F_0
- Truss element with large stiffness and large force F_0

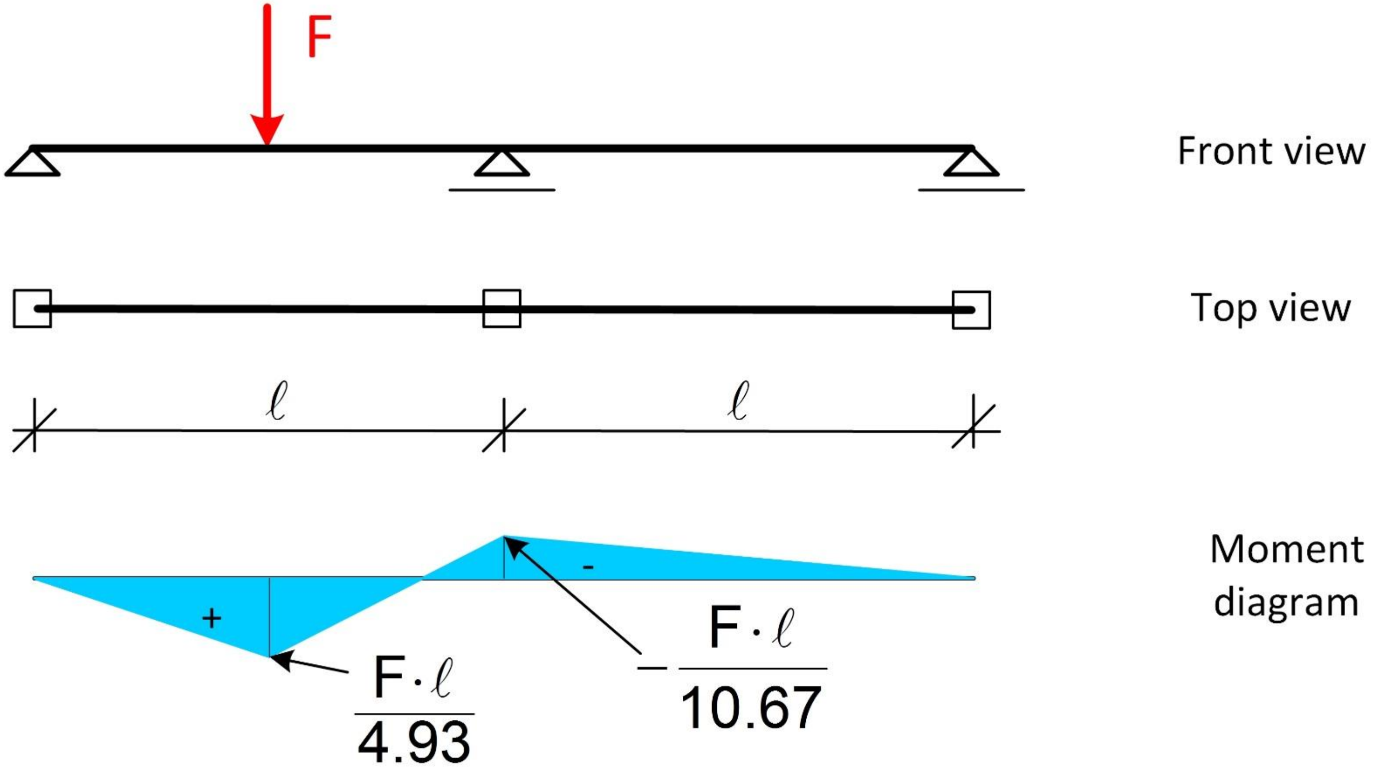
k_0 must be chosen such that the load F_0 is transfered almost totally to the spring.

Otherwise, k_0 and F_0 have to be augmented!

Modeling of supports

Example: Influence of support conditions

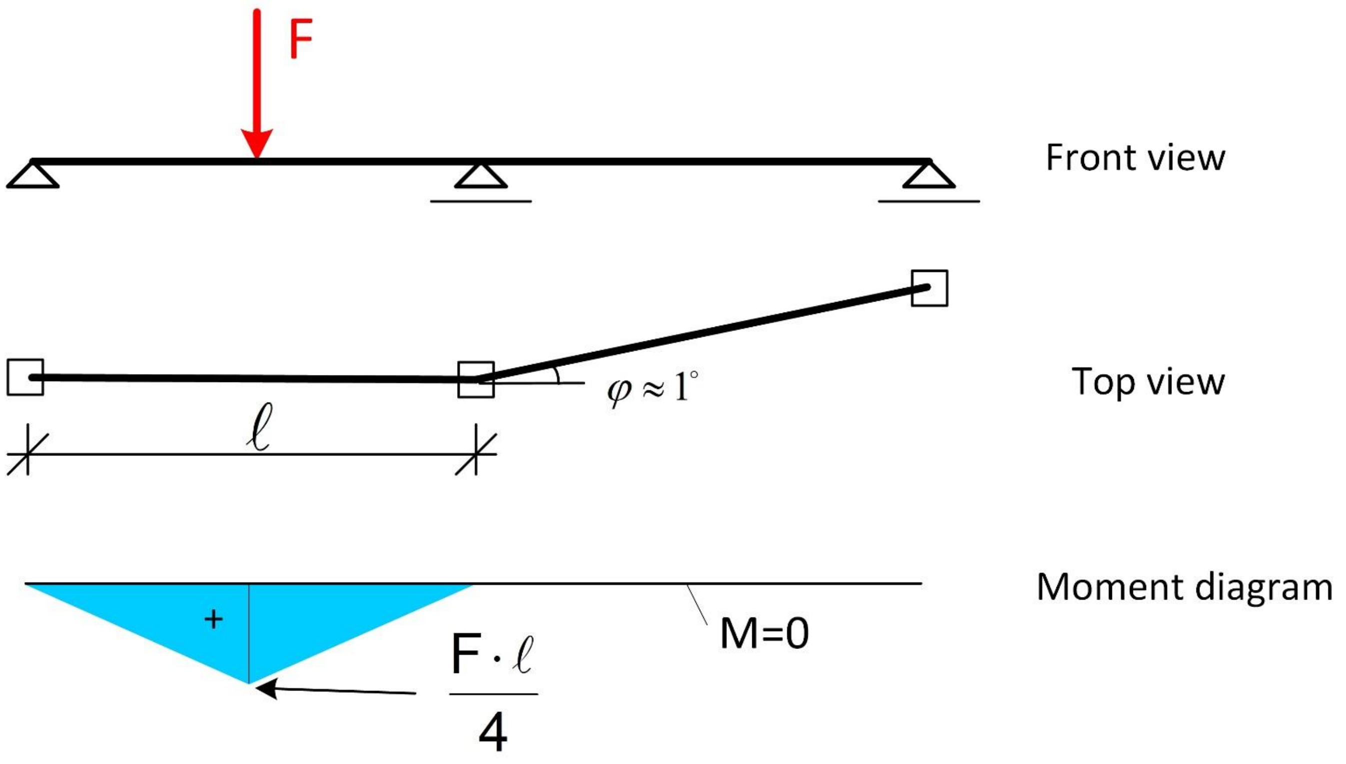
Case 1: 2D-Model of a continuous beam



Modeling of supports

Example: Influence of support conditions

Case 2: All side simply supported 3D-beam

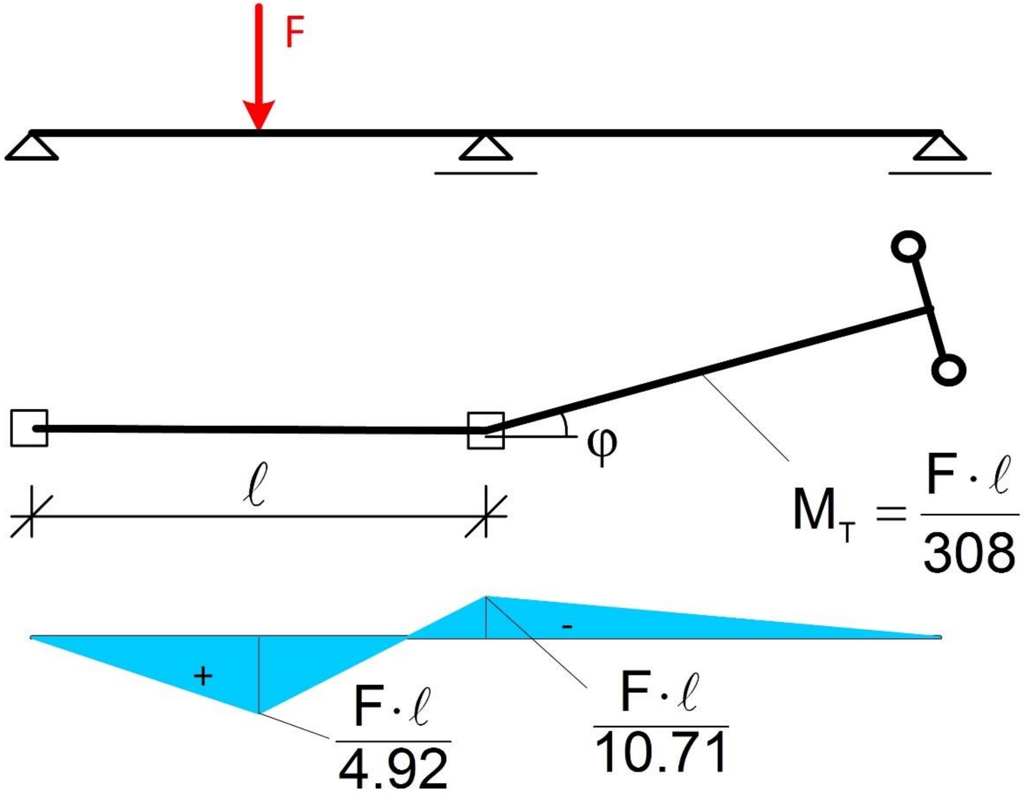


Moment equilibrium

Modeling of supports

Example: Influence of support conditions

Case 3: 3D-Model of a continous beam with torsion restraint at the right bearing

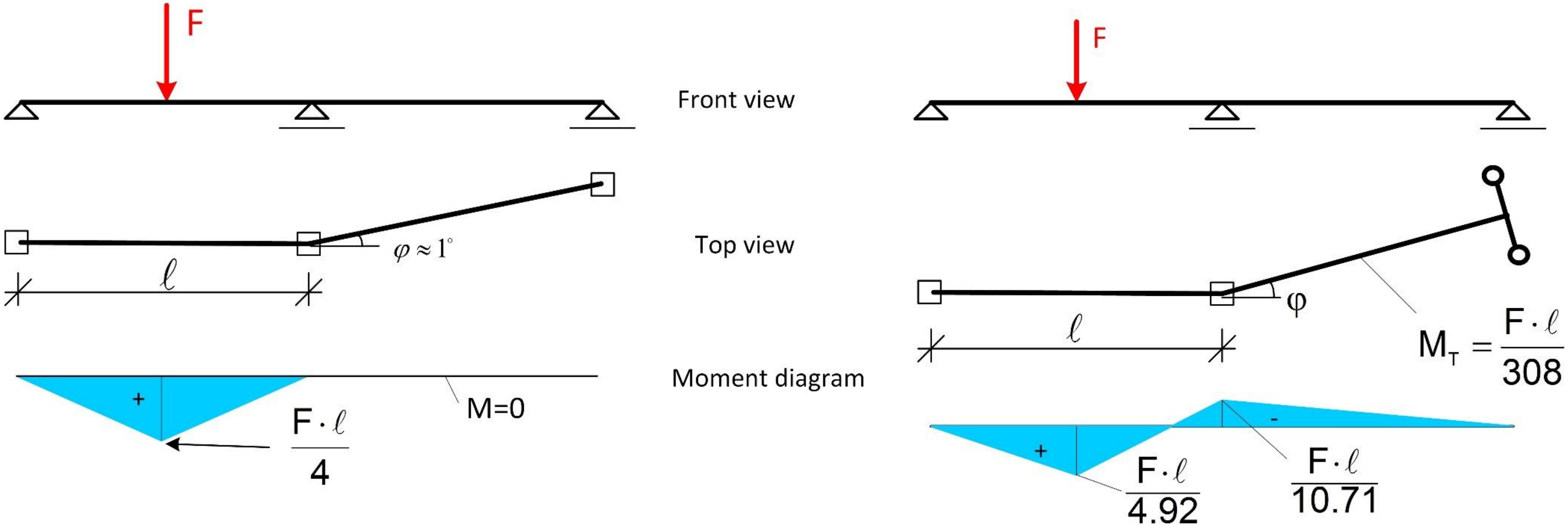


For an angle **φ≠0** a fixing of the torsional degree of freedom of the right beam is required in order to transfer the bending moment M_{li} !

Modeling of supports

Example: Influence of support conditions

Result:

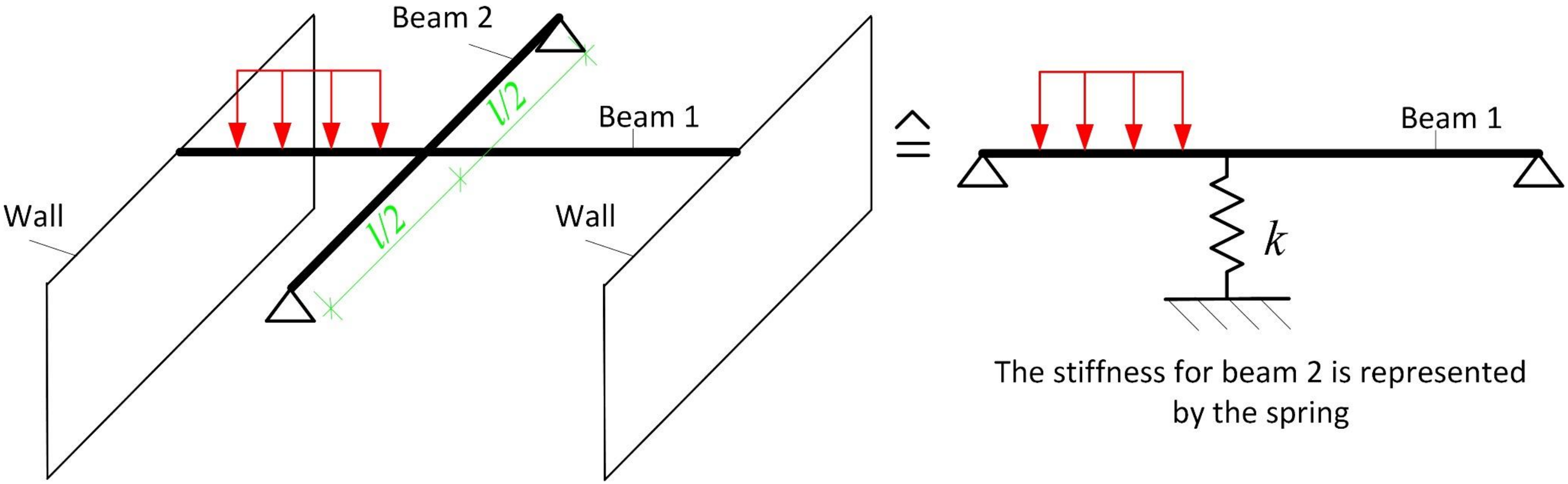


3-D systems with torsional degrees of freedom must be modeled carefully !

Modeling of springs

Displacement and rotational springs allow the modeling of
any elastic flexibility of supports and elastic restraints.

Example: Modeling of a simply supported beam as equivalent spring

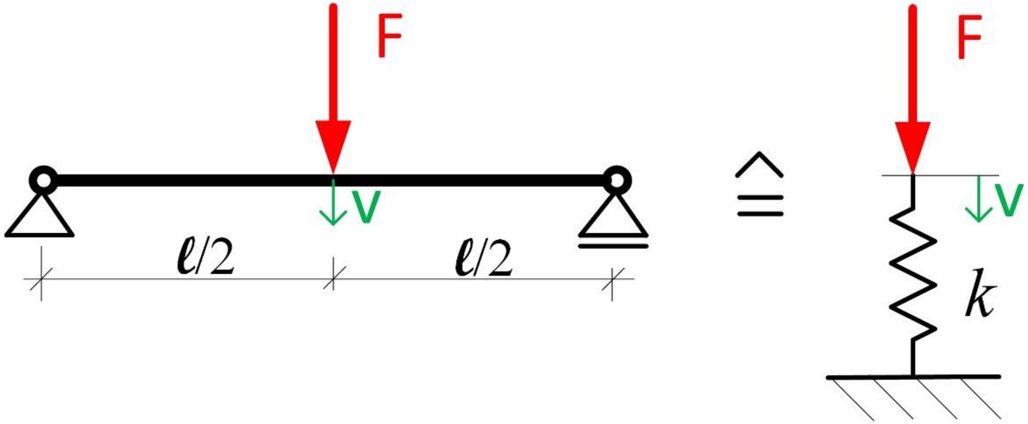


The stiffness for beam 2 is represented
by the spring

Instead of the 3D system, only a 2D-system has to be analysed.

Modeling of springs

Example: **Modeling of a simply supported beam as an equivalent spring**

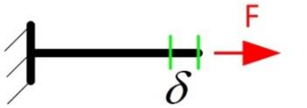
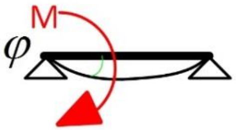
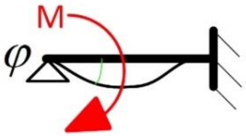
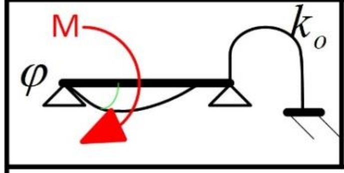


$$F = k \cdot v$$

$$v = \frac{F \cdot \ell^3}{48 \cdot E \cdot I} \quad \Rightarrow \quad F = \frac{48 \cdot E \cdot I}{\ell^3} \cdot v \quad \Rightarrow \quad k = \frac{48 \cdot E \cdot I}{\ell^3}$$

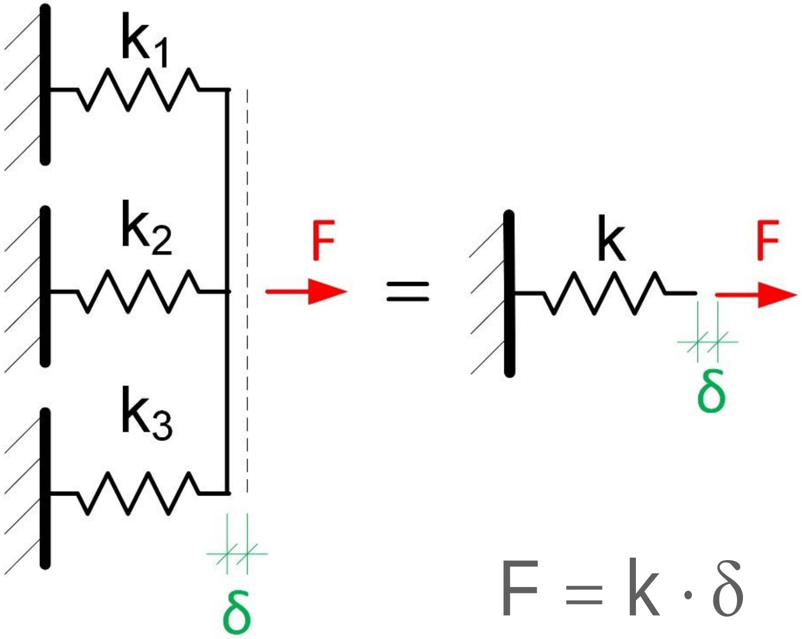
Modeling of springs

Replacement of adjacent beams by equivalent spring elements

System		Spring constant
	$F = k_v \cdot \delta$	$k_v = \frac{E \cdot A}{\ell}$
	$M = k_\varphi \cdot \varphi$	$k_\varphi = \frac{3 \cdot E \cdot I}{\ell}$
	$M = k_\varphi \cdot \varphi$	$k_\varphi = \frac{4 \cdot E \cdot I}{\ell}$
	$M = k_\varphi \cdot \varphi$	$k_\varphi = \frac{4 \cdot \ell \cdot k_o + 12 \cdot E \cdot I}{4 \cdot \ell + \ell^2 \cdot k_o / E \cdot I}$
EA = Longitudinal stiffness EI = Bending stiffness		ℓ = Length of a beam k_φ = Rotational spring constant

Modeling of springs

Springs in parallel



$$F = k \cdot \delta$$

Forces in individual springs:

$$F_1 = \frac{k_1}{k} \cdot F$$

$$F_2 = \frac{k_2}{k} \cdot F$$

$$F_3 = \frac{k_3}{k} \cdot F$$

Spring constant:

$$k = k_1 + k_2 + k_3$$

Condition: All springs have the same displacement

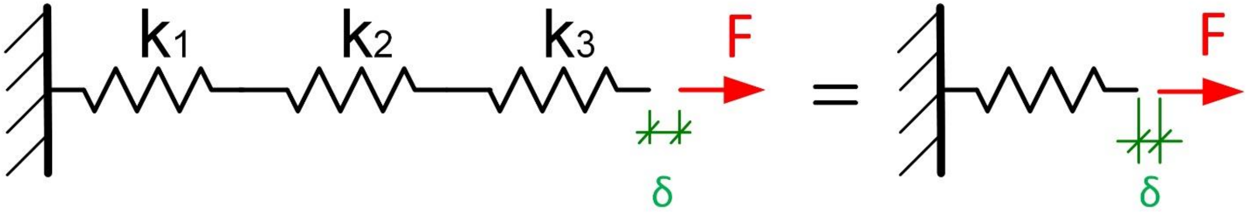
Derivation:

$$F = F_1 + F_2 + F_3 \qquad F_1 = k_1 \cdot \delta \qquad F_2 = k_2 \cdot \delta \qquad F_3 = k_3 \cdot \delta$$

$$k \cdot \delta = k_1 \cdot \delta + k_2 \cdot \delta + k_3 \cdot \delta \quad \Rightarrow \quad k = k_1 + k_2 + k_3 \qquad with \;\; \delta = F / k$$

Modeling of springs

Springs in series



Forces in individual springs:

$$F_1 = F_2 = F_3 = F$$

Spring constant:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

$$F = k \cdot \delta$$

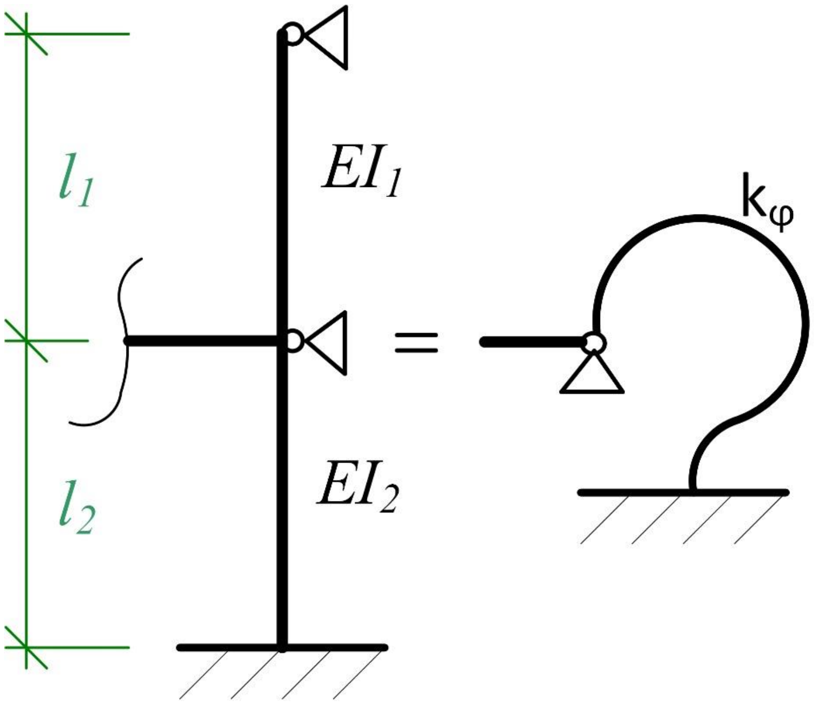
Condition: All springs have the same force

Derivation: $\delta = \delta_1 + \delta_2 + \delta_3$ $\quad \delta_1 = F/k_1 \quad \delta_2 = F/k_2 \quad \delta_3 = F/k_3$

$$F/k = F/k_1 + F/k_2 + F/k_3 \quad \Rightarrow \quad 1/k = 1/k_1 + 1/k_2 + 1/k_3$$

Modeling of springs

Example: Elastic restraint of a beam



Real system

Equivalent system

The bending stiffnesses of beams 1 and 2 are represented by rotational springs:

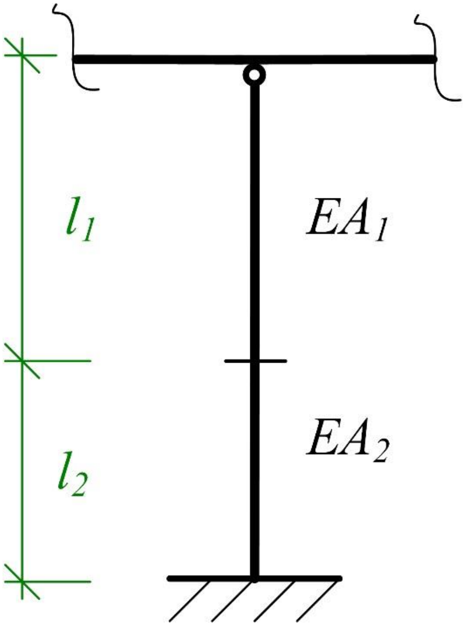
$$k_{\varphi 1} = \frac{3 \cdot E \cdot I_1}{\ell_1} \qquad k_{\varphi 2} = \frac{4 \cdot E \cdot I_2}{\ell_2}$$

Springs in parallel

$$k_\varphi = k_{\varphi 1} + k_{\varphi 2} = \frac{3 \cdot E \cdot I_1}{\ell_1} + \frac{4 \cdot E \cdot I_2}{\ell_2}$$

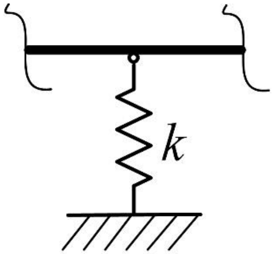
Modeling of springs

Example: Column with variable cross section



Real system

=



Equivalent system

The stiffness of the column is represented by springs:

$$k_1 = \frac{E \cdot A_1}{\ell_1} \qquad k_2 = \frac{E \cdot A_2}{\ell_2}$$

Springs in series:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{\ell_1}{E \cdot A_1} + \frac{\ell_2}{E \cdot A_2}$$

$$k = \frac{E \cdot A_1 \cdot A_2}{A_2 \cdot \ell_1 + A_1 \cdot \ell_2}$$

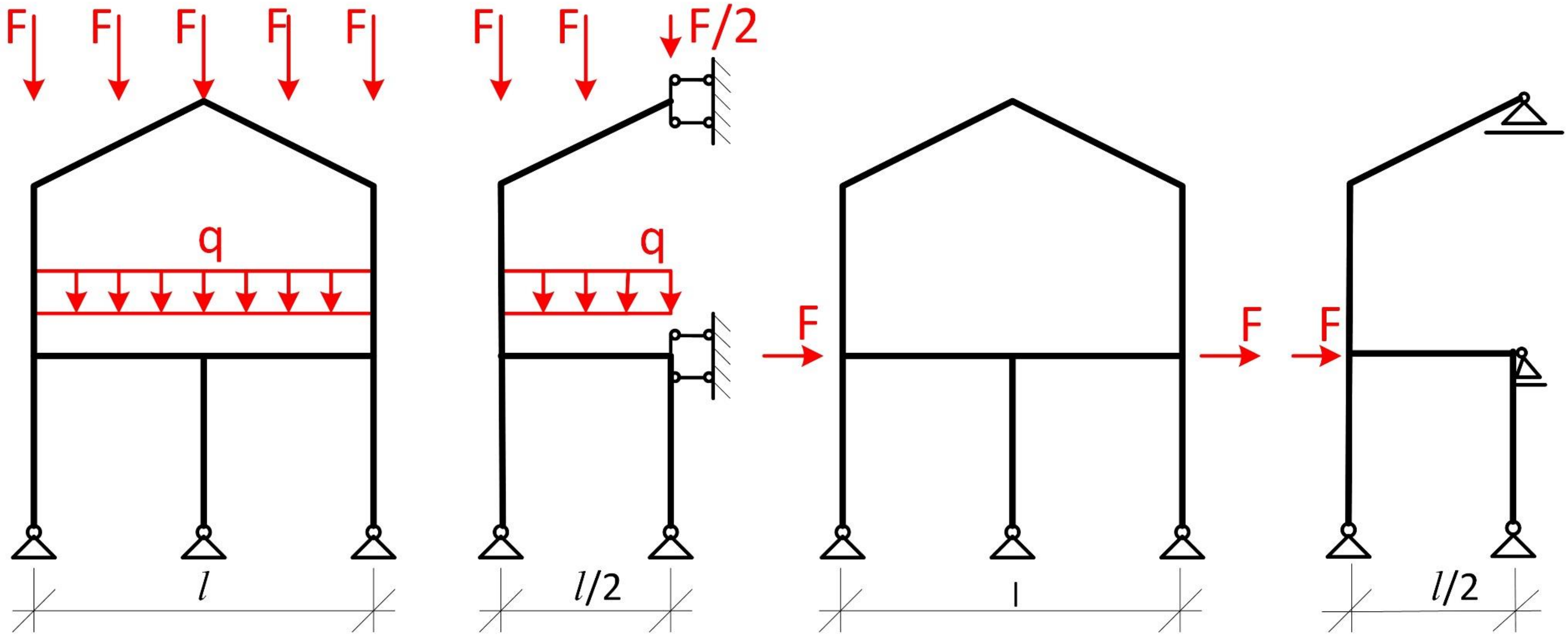
Symmetric systems

The computational effort for large symmetrical finite element systems can be reduced by taking advantage of the symmetry of the system.

Example: Symmetric frames

symmetric loading

anti-symmetric loading



System

Equivalent system

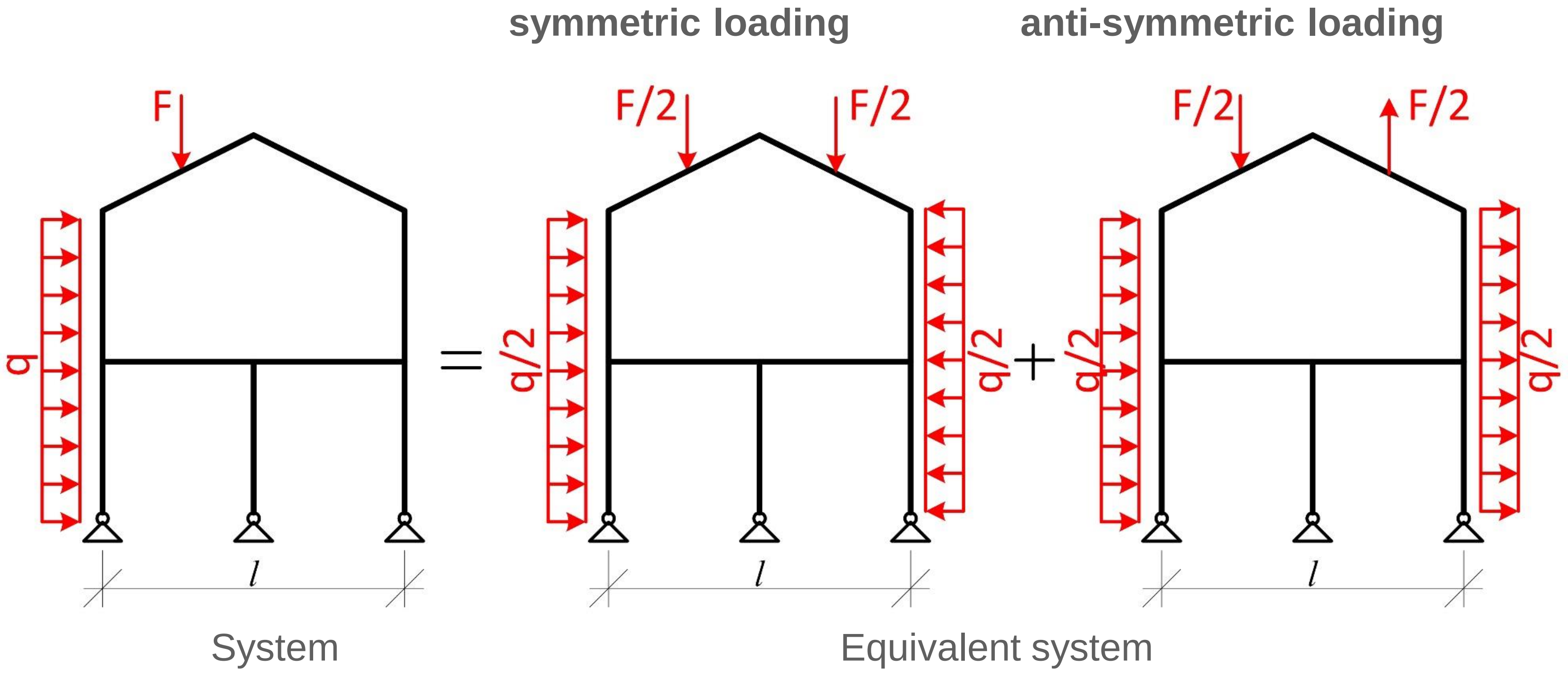
System

Equivalent system

Symmetric systems

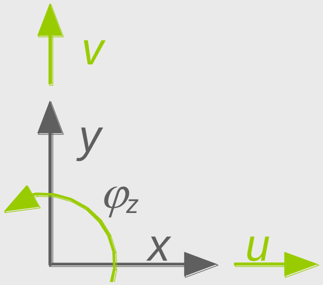
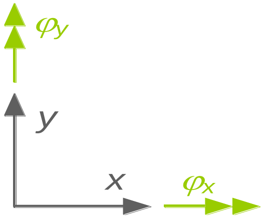
The computational effort for large symmetrical finite element systems can be reduced by taking advantage of the symmetry of the system.

Example: **Symmetric frames**



Symmetric systems

Symmetry conditions of plane symmetrical systems

System	Loading	
	symmetric	**anti-symmetric**
 Frame / Plate in Plane stress	 **u=0** **v=0** $\varphi_z = 0$ $\varphi_z = 0$	 **v=0** **u=0**
 Girder grid / Plate in bending	 $\varphi_y = 0$ $\varphi_x = 0$	 **w=0** **w=0** $\varphi_x = 0$ $\varphi_y = 0$

Symmetric systems

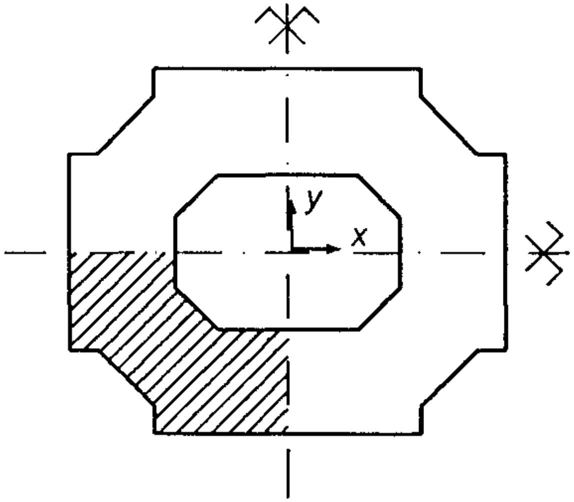
Stresses and section forces in symmetrical systems

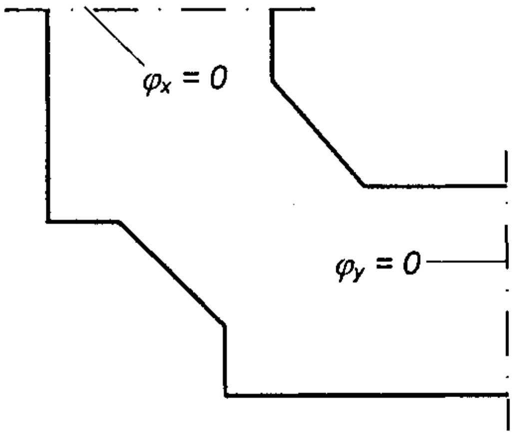
System	Section force/ Stress component	Loading	
		symmetric	anti-symmetric
Beam structures	• Bending moments • Normal forces • Torsional moments • Deflection	symmetric	anti-symmetric
	• Shear forces	anti-symmetric	symmetric
Plates in plane stress	• Normal stresses • Deflection	symmetric	anti-symmetric
	• Shear stresses	anti-symmetric	symmetric
Plates in bending	• Bending moments • Shear forces • Deflection	symmetric	anti-symmetric
	• Twisting moments	anti-symmetric	symmetric

Symmetric systems

Example: Structural slab with several symmetry axes

Partial system of one quarter of the plate





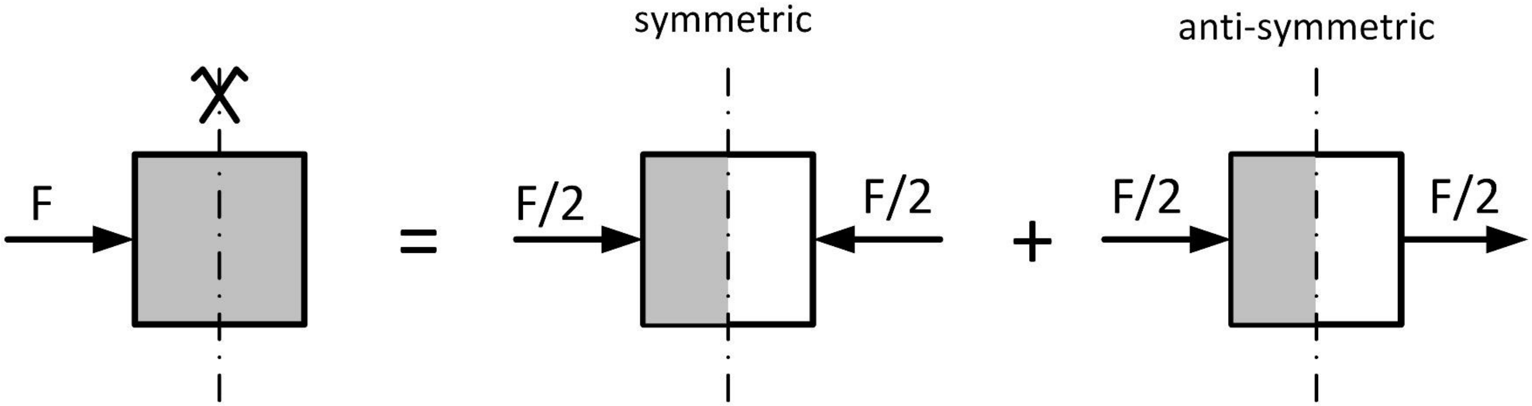
Structural slab

Equivalent partial system

The system could be further simplified to one-eighth of the actual system. However in this case inclined supports have to be defined.

Symmetric systems

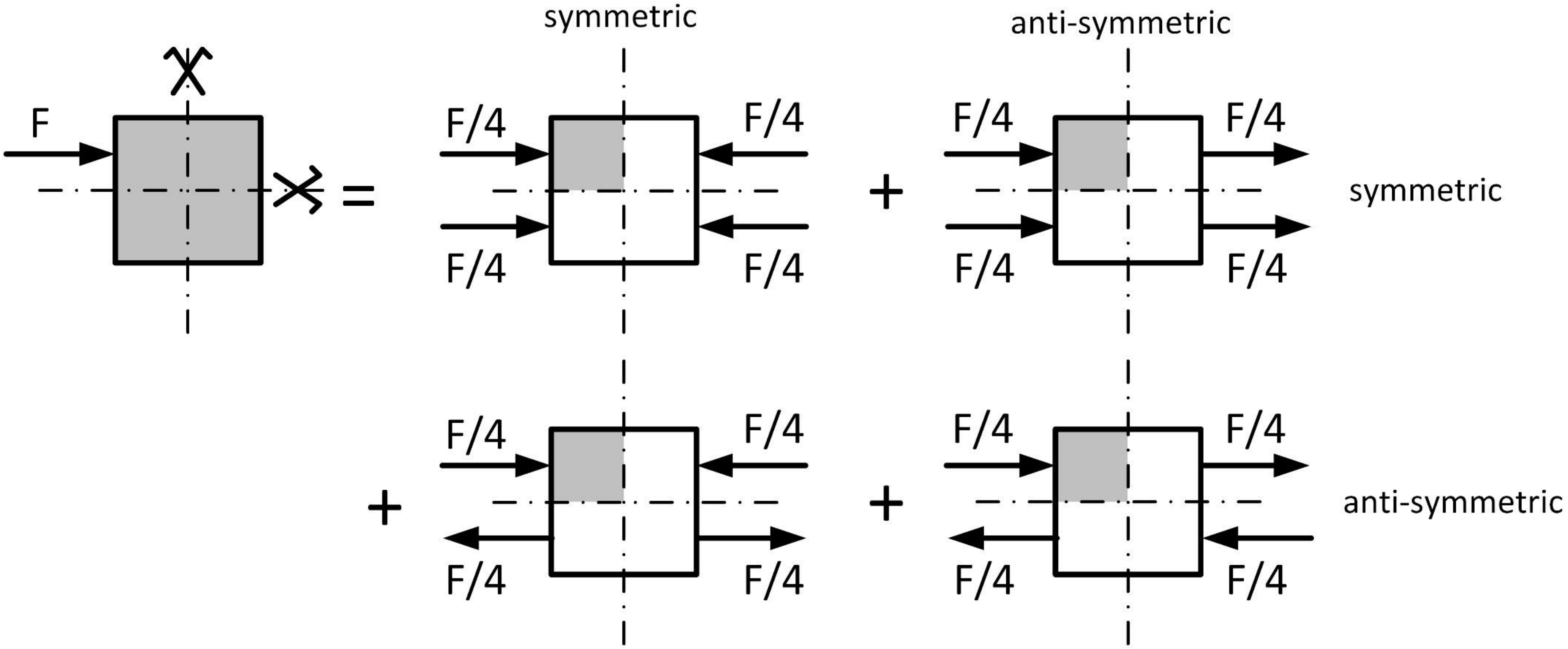
Example: Superposition for anti-symmetric loads



System with one axis of symmetry

Symmetric systems

Example: Superposition for anti-symmetric loads



System with two axes of symmetry

Quality assurance of finite element analyses of beam systems

Sources of errors

Types of error:

- Error in the computational model
- Input error
- Numerical error
- Program error

Detectable errors of the program:

- inconsistent input data
- kinematic mechanisms
- physically meaningless material and cross section parameters

Most errors cannot be detected by the FE program!

Quality assurance of finite element analyses of beam systems

Possible sources of errors

Input errors

Causes:

- Inattention
- Misunderstanding of the program manual
- Misleading information in the program manual.

Remedies:

- Plausibility control of the solutions
- Conclusive careful inspection of all statically relevant input data.

The most frequent type of error!

Quality assurance of finite element analyses of beam systems

Possible sources of errors

Program error

Errors in the program code:	Rare, but can never be excluded!
	Especially critical are rarely used program functions

Example

A program for the analysis of frame structures reveals a program bug when the load case of a moment on a beam as element load is calculated. In the previous version the program gave the correct result.

Moment applied as element load

Quality assurance of finite element analyses of beam systems

Possible sources of errors

Numerical errors

Numbers in computations have a *finite* numerical accuracy.

Example: **Pocket calculator**

$1000 + 1 - 1000 = 1$ ok
$10^{20} + 1 - 10^{20} = 0$ error, i.e. the calculator has less than 20 digits mantissa accuracy
$10^{9} + 1 - 10^{9} = 1$
$10^{10} + 1 - 10^{10} = 0$ The calculator used has 9 digits mantissa accuracy

When assembling the element stiffness matrices into the global stiffness matrix, all significant digits may be lost in the case of vastly different terms of the element stiffness matrices!

Quality assurance of finite element analyses of beam systems

Possible sources of errors

Numerical errors

Cause: Limited computational accuracy of the computer
Numerical values are represented in the computer by a mantissa and exponent
Both values are limited due to the limited computer storage capacity.

Example: The number π is represented in single precision (6 digits of the mantissa) as:

Exact: $\pi = 3.1415927...$

With single precision : $\pi = \underset{\substack{\uparrow \\ \text{Mantissa}}}{0.314159} \cdot 10^{\underset{\substack{\uparrow \\ \text{Exponent}}}{1}}$

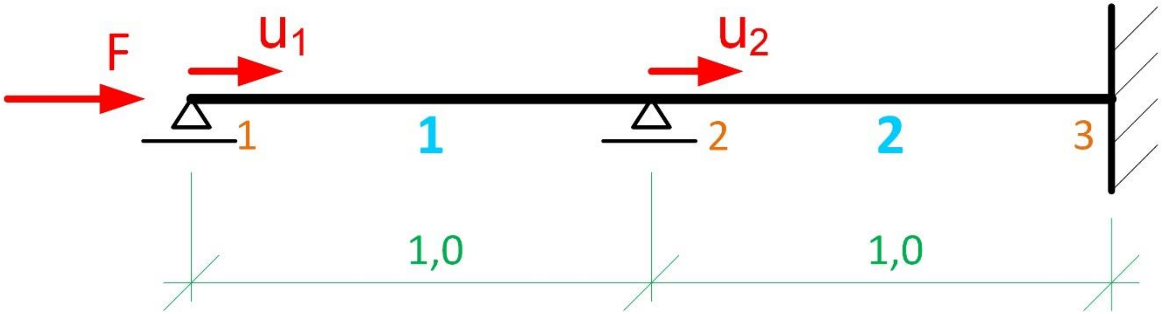
All other digits are omitted when the number is stored in the computer.

Number representation

Quality assurance of finite element analyses of beam systems

Example: Extreme differences in stiffness

To represent a rigid coupling between nodal point 1 and nodal point 2, the element stiffness matrix of element 1 is chosen to be extremely large.



Element stiffnesses:

Element 1: EA = 10^{20}

Element 2: EA = 10^{5}

Element stiffness matrices

Truss element 1:

$$10^{20} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10^{20} & -10^{20} \\ -10^{20} & 10^{20} \end{bmatrix}$$

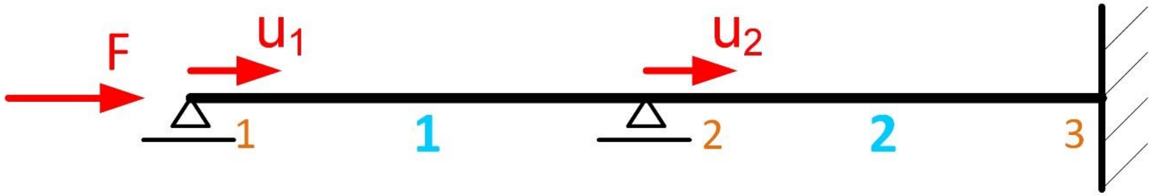
Truss element 2:

$$10^{5} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10^{5} & -10^{5} \\ -10^{5} & 10^{5} \end{bmatrix}$$

Quality assurance of finite element analyses of beam systems

Example: Extreme differences in stiffness

Global stiffness matrix



$$\begin{bmatrix} 10^{20} & -10^{20} & 0 \\ -10^{20} & 10^{20}+10^{5} & -10^{5} \\ 0 & -10^{5} & 10^{5} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ F_3 \end{bmatrix}$$

with $u_3 = 0$ →

$$\begin{bmatrix} 10^{20} & -10^{20} \\ -10^{20} & 10^{20}+10^{5} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Global stiffness matrix for the system if numerical values are represented with 15 digits accuracy: →

$$\begin{bmatrix} 10^{20} & -10^{20} \\ -10^{20} & 10^{20} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Singular global stiffness matrix!

Mathematical: Singular global stiffness matrix

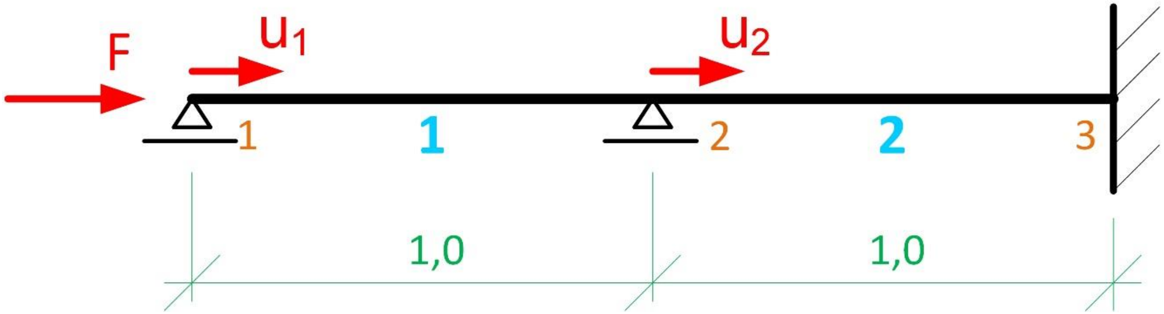
Statical: Element 2 is neglected in structural analysis, i.e. the system is kinematic

Quality assurance of finite element analyses of beam systems

Example: Extreme differences in stiffness

This effect does not occur if the extremely rigid element will be added to the diagonal, e.g. if it represents a spring.

Example: Element stiffness 2 will be chosen to be extremely large.



Element stiffnesses:

Element 1: EA = 10^5

Eleemnt 2: EA = 10^{20}

Element stiffness matrices

Truss element 1:

$$10^5 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10^5 & -10^5 \\ -10^5 & 10^5 \end{bmatrix}$$

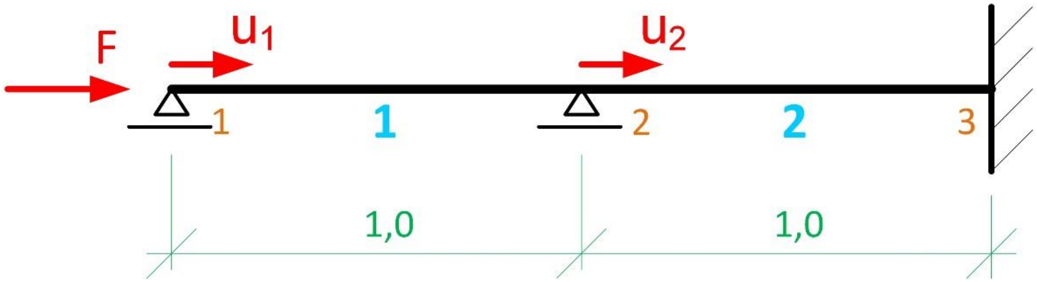
Truss element 2:

$$10^{20} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10^{20} & -10^{20} \\ -10^{20} & 10^{20} \end{bmatrix}$$

Quality assurance of finite element analyses of beam systems

Example: Extreme differences in stiffness

Global stiffness matrix



$$\begin{bmatrix} 10^5 & -10^5 & 0 \\ -10^5 & 10^5+10^{20} & -10^{20} \\ 0 & -10^{20} & 10^{20} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ F_3 \end{bmatrix}$$

with $u_3 = 0$ $\rightarrow$

$$\begin{bmatrix} 10^5 & -10^5 \\ -10^5 & 10^5+10^{20} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Global stiffness matrix for the system if numerical values are represented with 15 digits accuracy: $\rightarrow$

$$\begin{bmatrix} 10^5 & -10^5 \\ -10^5 & 10^{20} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Regular global stiffness matrix!

Mathematical: Regular global stiffness matrix

Statical: Element 2 is taken into account, i.e. the system is not kinematic

Quality assurance of finite element analyses of beam systems

Numerical errors

Example: Extreme differences in stiffness

- Extremly large stiffness differences between neighboring elements must be avoided.
- At the boundary of the finite element system, extremely large siffnesses, e.g. of spring constants, are unproblematic.

Cause

- extreme differences in the stiffness of neighboring elements, e.g. $E \cdot A,\ E \cdot I,\ G \cdot I_T$
- extreme differences in the element lengths

Remedies

- Rigid couplings (if available in the program)
- Limitation of the stiffness differences to a physically meaningful measure

Quality assurance of finite element analyses of beam systems

Verification

- Every finite element analysis requires an intensive check of its correctness.
- The aim is to avoid sources of error.
- Most important is a critical survey of the program input and output.

Types of error:

- Error in the computational model
- Input error
- Numerical error
- Program error

Quality assurance of finite element analyses of beam systems

Verification

Verification of a FE analysis

- **Detection of gross errors:** careful examination of the graphical display of the structural system, its deformations and section forces (eg. check of missing supports, wrong sign of the loading, etc.)

- **Sum-of-the-loads check:** simple hand calculation which may give useful hints in case of gross loading input errors.

- **Plausibility check of force and moment diagrams:** simple hand calculations on simplified partial models. Check of the equilibrium conditions of the support forces and the total loading.

- **Final careful examination of all statically relevant input data.**

Quality assurance of finite element analyses of beam systems

Verification

Checking in case of termination due to a singular stiffness matrix

- Program termination without any output of results -

- Examination of the support conditions
- Examination for kinematic mechanisms caused by single hinges or by the combination of hinge conditions
- Examination of free nodal points with degrees of freedom not connected with any element
- Examination of the longitudinal, bending, and torsional stiffness of all elements (must be unequal to zero)
- Examination of extreme jumps in the stiffness of neighbouring elements, causing numerical errors

Quality assurance of finite element analyses of beam systems

Verification

Checks in the case of doubts about the correctness of the results

A stepwise simplification of the system where the remaining partial system contains the supposed error is a good strategy. Finally a simple system should remain where the error is obvious.

Most extensive control

If there remain doubts about the correctness of the results after a careful examination of the program input and results, a new independent analysis with a different program and by a different person is recommended.

End

Introduction

2 Truss and beam structures

Plate and shell structures

Modeling

Quality assurance of finite element analyses of beam systems

Programing language	Type of number	Digits of the Mantissa	Max. exponent
Fortran	real*4 real*8	6 15	≈ 37 ≈ 307
C/C++ **Java**	float double	6 15	≈ 37 ≈ 307

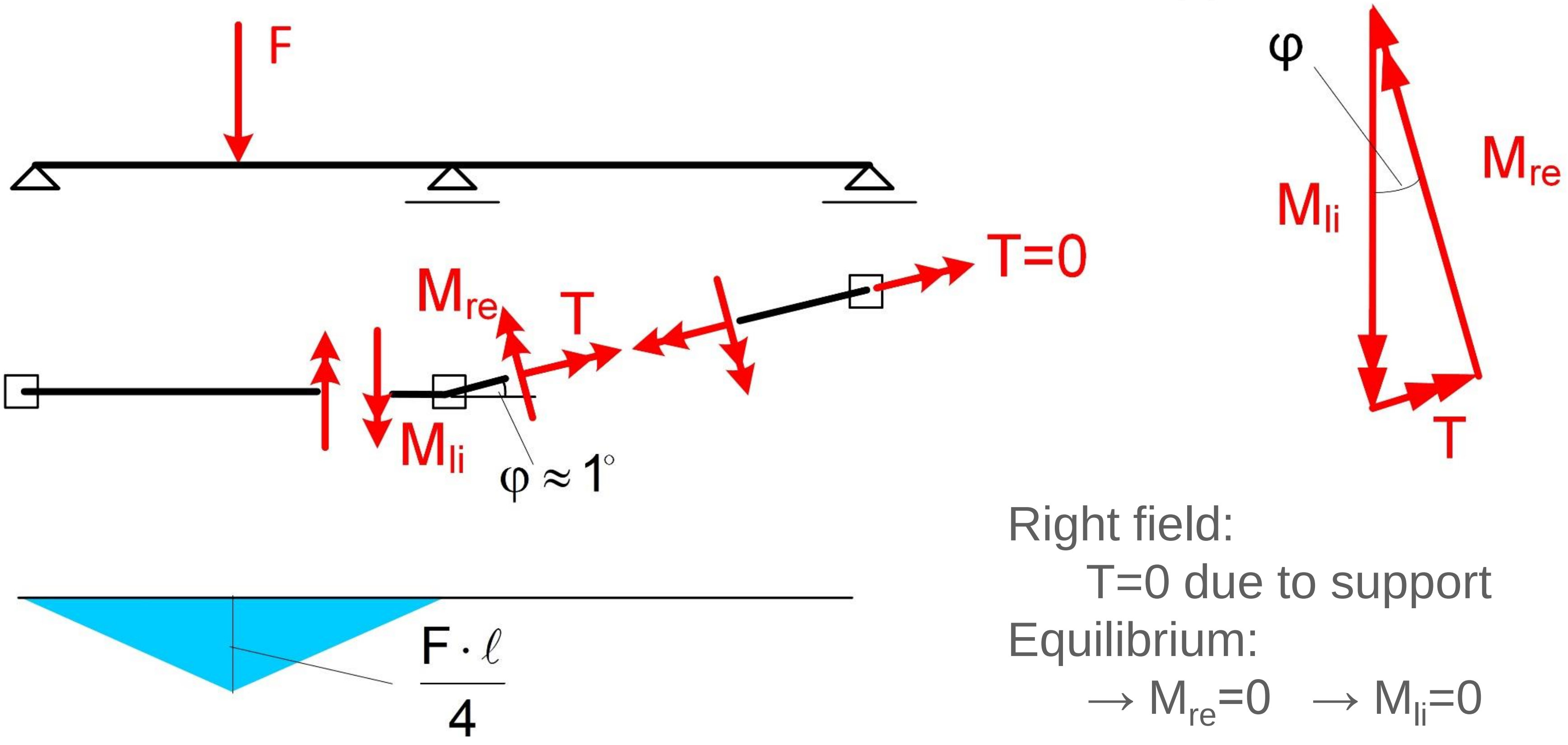
Representation of floating point numbers in the computer

Modeling of supports

Example: Influence of support conditions

Case 2: All side simply supported 3D-Model

Moment equilibrium at the center support



Right field:

T=0 due to support

Equilibrium:

→ M_{re}=0 → M_{li}=0