Finite Elements in Structural Analysis

Introduction

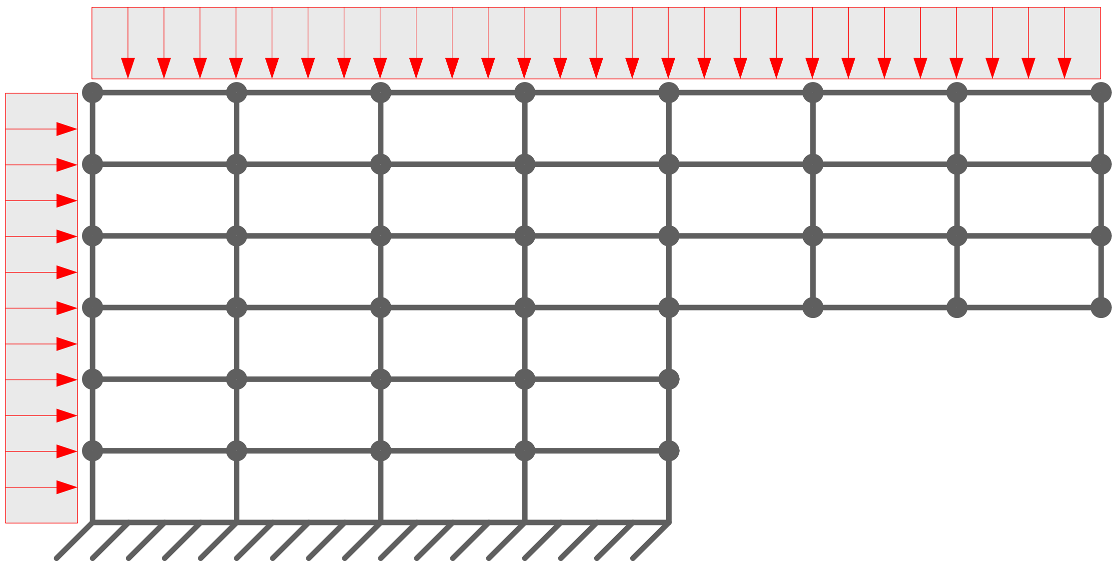
Truss and beam structures

3 Plate and shell structures

Modeling

Finite element method for plate and shell structures

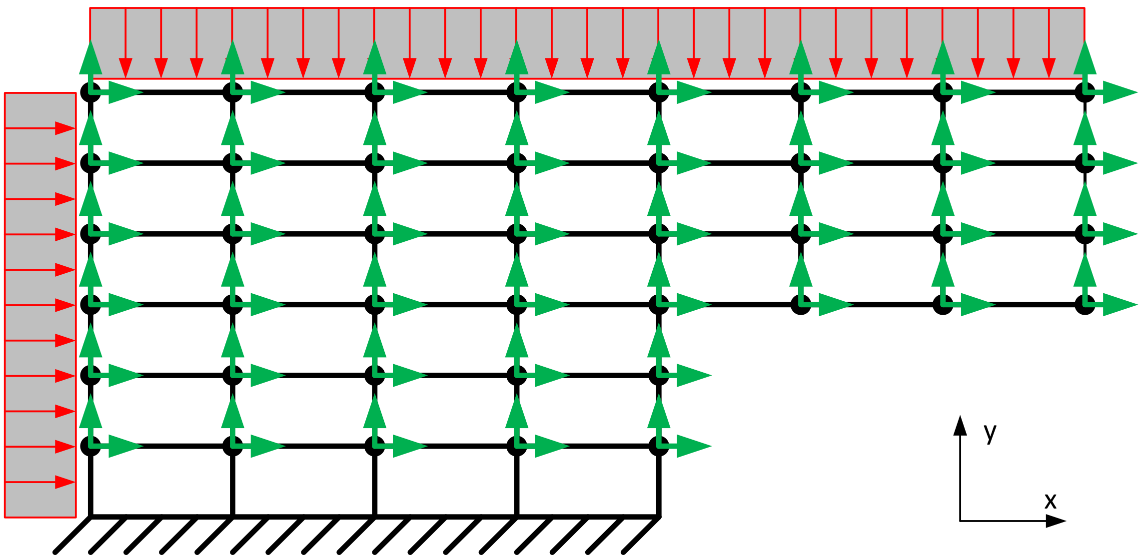
Plate in plane stress



- Plate discretization in elements of finite size (e.g. quadrilateral elements with ~ 1 m side length).
- The elements are connected at the nodal points.

Finite element method for plate and shell structures

Plate in plane stress

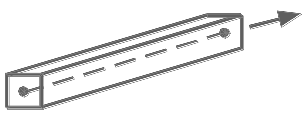


System of equations

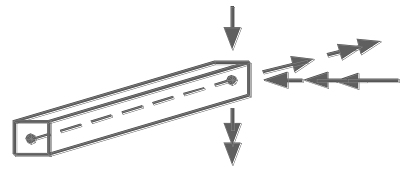
42 nodes with 2 degrees of freedom each ➡ 84 equations

Finite element method for plate and shell structures

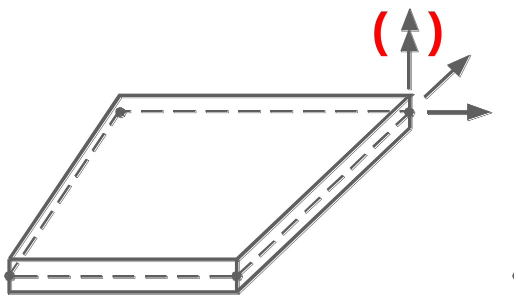
Element types and degrees of freedom



Truss element



Beam in bending



plane stress element

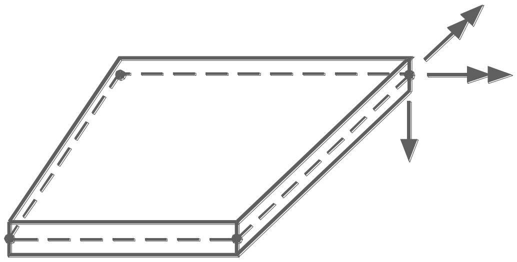
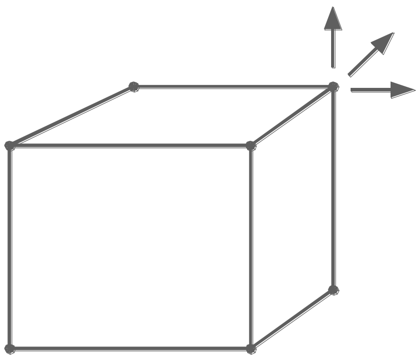
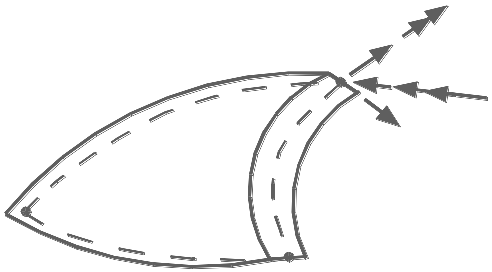


Plate element



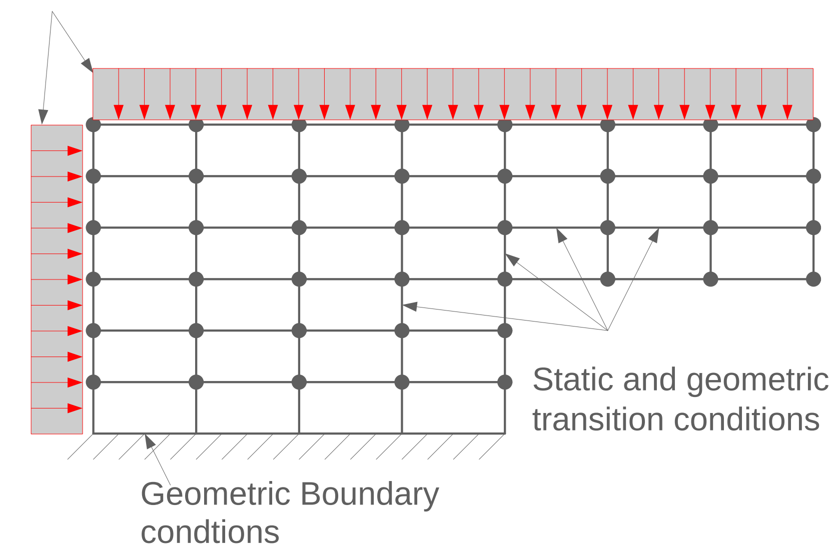
3D-continuum element



Shell element

Finite element method for plate and shell structures

Boundary and transition conditions



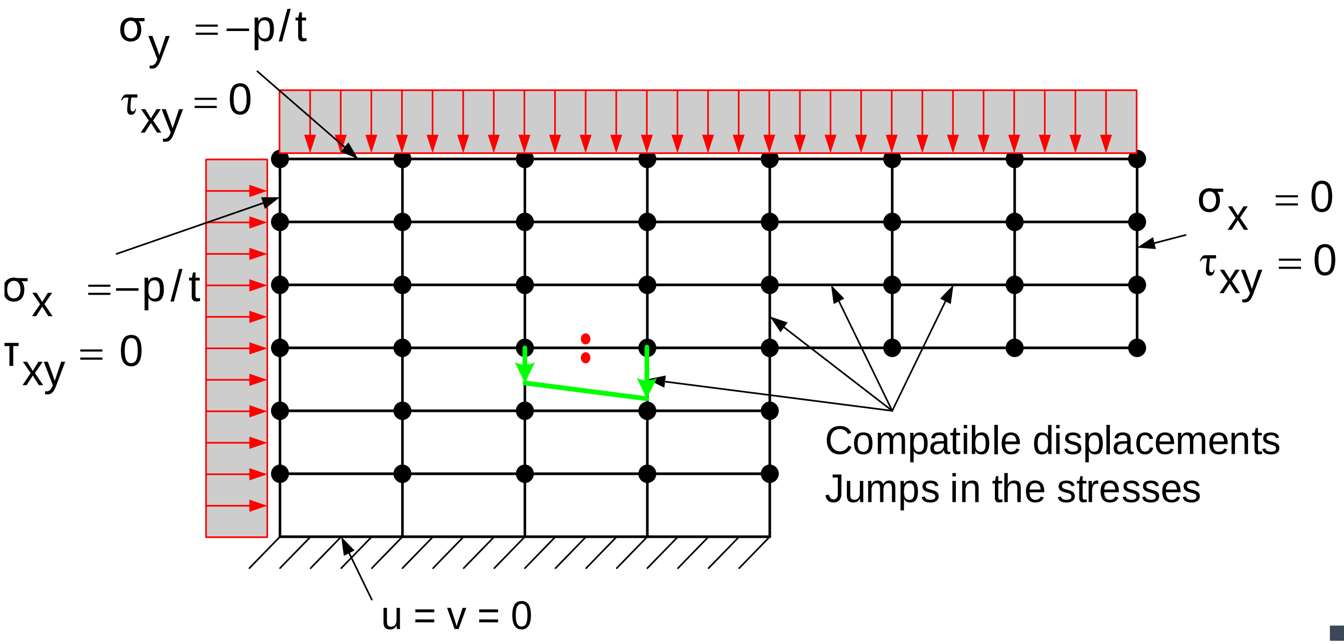
Compatibility conditions at the boundaries of adjacent elements

Condition	FEM
1. Compabitility of the displacements between the nodal points	**fulfilled**
2. Compatibility of the stresses at the element boundaries (Equilibrium conditions)	**not fulfilled**

Finite element method for plate and shell structures

Boundary and transition conditions

Example: Two adjacent elements



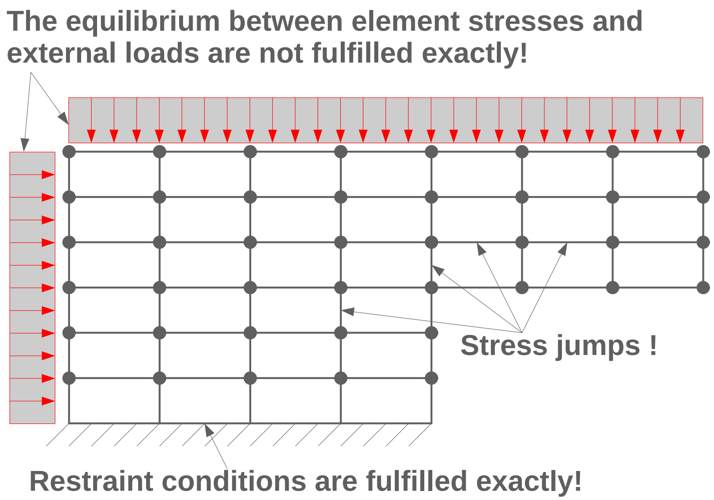
Displacements
At their common boundary both elements have the same displacements (varying linearly between the nodal points)

Stresses
The upper element has different stresses σ_x, σ_y, τ_{xy} than the lower element.

➡ Violation of the equilibrium conditions !

Finite element method for plate and shell structures

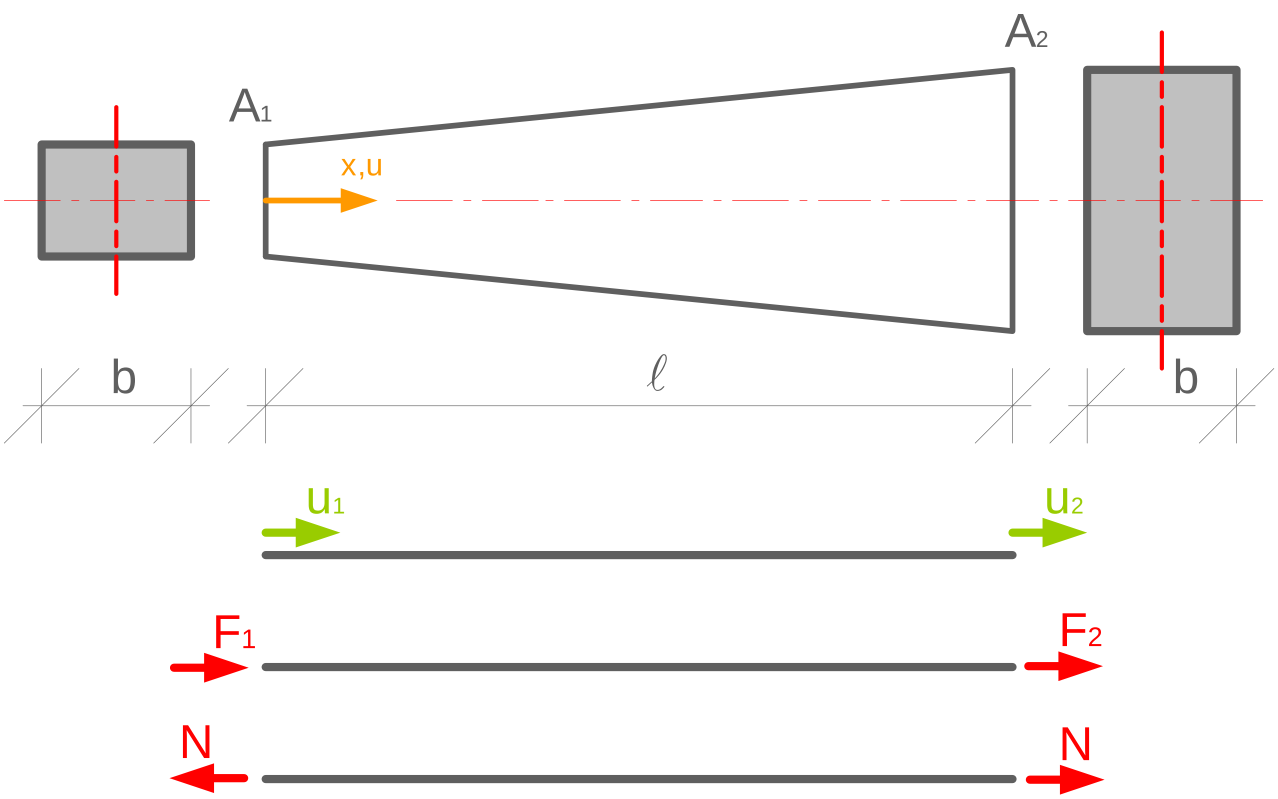
Properties of the displacement-based FEM



- Displacements of adjacent plane stress elements coincide at the boundaries.
- The equilibrium conditions for the stresses are not fulfilled exactly at the boundary lines, resulting in a nonrealistic „jump“ of the stesses or section forces between elements.
- Support conditions of the displacements are exactly fulfilled at fixed boundaries.
- At free boundaries the equilibrium conditions between the boundary loads and the section forces are not fulfilled exactly.

One-dimensional example

Truss element with variable cross section area



Truss element

Cross section area:

$$A = A_1 + \frac{x}{\ell} \cdot (A_2 - A_1)$$

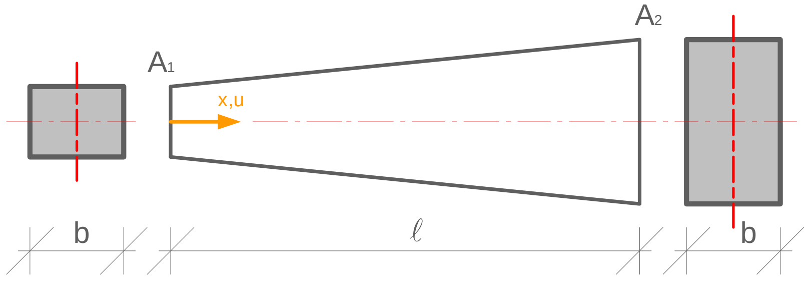
Nodal point displacements

Nodal forces

Normal force

Example: Analytical solution

Derivation:



$$A = A_1 + \frac{x}{\ell} \cdot (A_2 - A_1)$$

Normal stress σ_x:

$$\sigma_x = \frac{N}{A} = \frac{N}{A_1 + x/\ell \cdot (A_2 - A_1)}$$

$$\boxed{\sigma_x = \frac{N \cdot \ell}{A_1 \cdot \ell + x \cdot (A_2 - A_1)}}$$

Strain ε_x:

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{N \cdot \ell}{E \cdot A_1 \cdot \ell + x \cdot E \cdot (A_2 - A_1)}$$

Displacement u:

$$u = \int_0^x \varepsilon_x \, dx + u_1 = \int_0^x \frac{N \cdot \ell}{E \cdot (A_1 \cdot \ell + x \cdot (A_2 - A_1))} dx + u_1$$

$$\boxed{u = \frac{N \cdot \ell}{E (A_2 - A_1)} \cdot \ln\left(\frac{\ell \cdot A_1 + x (A_2 - A_1)}{\ell \cdot A_1} \right) + u_1}$$

Example: Analytical solution

Numerical example:

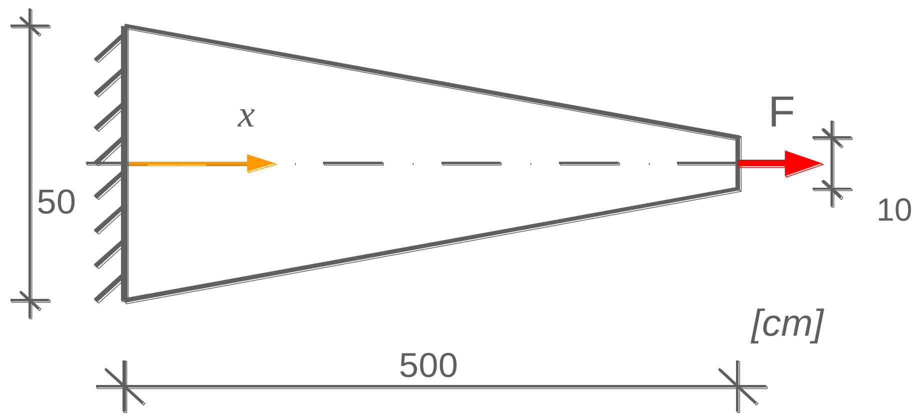
System

$A_1 = 500\ cm^2$

$A_2 = 100\ cm^2$

$E = 1000\ kN/cm^2$

$F = 100\ kN$



Displacements:

$$u = \frac{N \cdot \ell}{E(A_2 - A_1)} \cdot \ln\left(\frac{\ell \cdot A_1 + x(A_2 - A_1)}{\ell \cdot A_1}\right) + u_1 = \frac{100 \cdot 500}{1000 \cdot (100 - 500)} \cdot \ln\left(\frac{500 \cdot 500 + x \cdot (100 - 500)}{500 \cdot 500}\right) + 0$$

$$\boxed{u = -0.125 \cdot \ln(1 - 0.0016 \cdot x)}$$ with u [cm], x [cm]

Stresses:

$$\sigma_x = \frac{N \cdot \ell}{A_1 \cdot \ell + x \cdot (A_2 - A_1)} = \frac{100 \cdot 500}{500 \cdot 500 + x \cdot (100 - 500)}$$

$$\boxed{\sigma_x = \frac{100}{500 - 0.8 \cdot x}}$$ with x [cm] and σ_x [kN/cm²]

Example: Analytical solution

Numerical example:

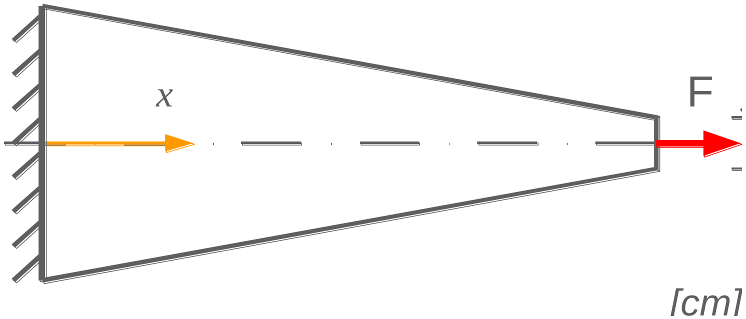
System

A_1 = 500 cm^2

A_2 = 100 cm^2

E = 1000 kN/cm^2

F = 100 kN



Displacements:

$$u = -0.125 \cdot \ln(1 - 0.0016 \cdot x)$$

Stresses:

$$\sigma_x = \frac{100}{500 - 0.8 \cdot x}$$

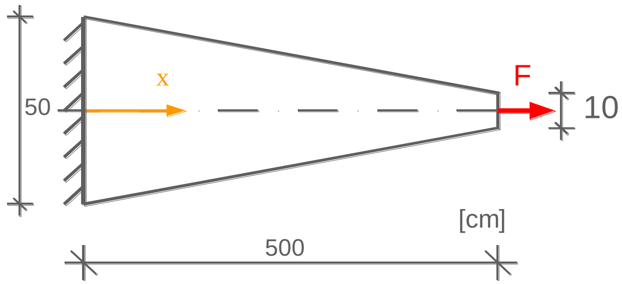
with x [cm], u [cm] and σ_x [kN/cm^2]

x [cm]	0	100	200	250	300	400	500
u [cm]	0	0.022	0.048	0.064	0.082	0.128	0.201
σ_x [kN/cm^2]	0.200	0.238	0.294	0.333	0.385	0.556	1.000

Example: Analytical solution

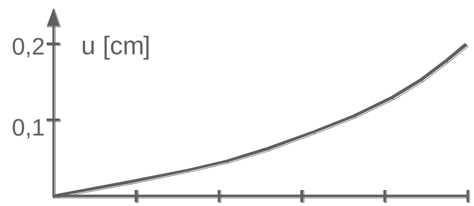
Numerical example:

System



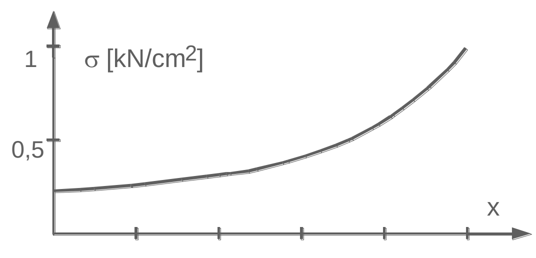
A_1 = 500 cm^2
A_2 = 100 cm^2
E = 1000 kN/cm^2
F = 100 kN

Displacements



$$u = -0.125 \cdot \ln(1 - 0.0016 \cdot x)$$

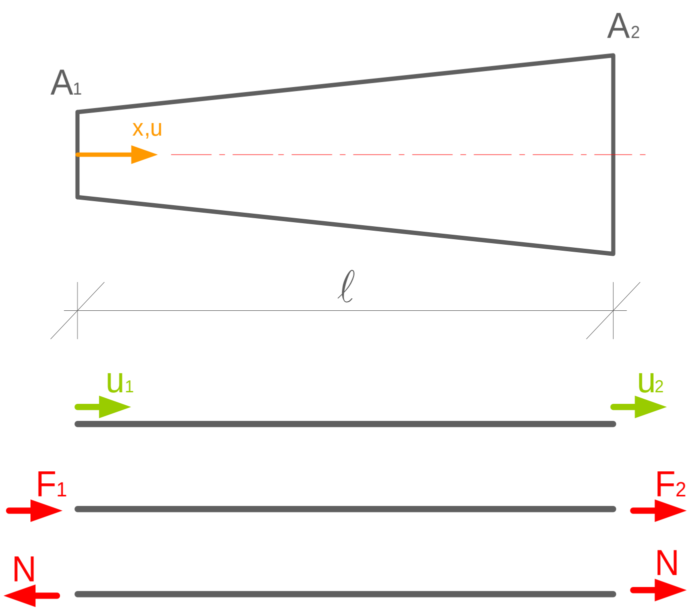
Stresses



$$\sigma_x = \frac{100}{500 - 0.8 \cdot x}$$

Example: Analytical solution

Stiffness matrix:



Displacements: $u = \frac{N \cdot \ell}{E(A_2 - A_1)} \cdot \ln\left(\frac{\ell \cdot A_1 + x(A_2 - A_1)}{\ell \cdot A_1}\right) + u_1$

Displacement u_2 at the end of the element $x = \ell$:

$$u_2 = u_{(x=1)} = \frac{N \cdot \ell}{E(A_2 - A_1)} \cdot \ln\left(\frac{A_2}{A_1}\right) + u_1 \quad \Rightarrow \quad N = \frac{E(A_2 - A_1)}{\ell \cdot \ln(A_2 / A_1)} \cdot (u_2 - u_1)$$

Equilibrium conditions:

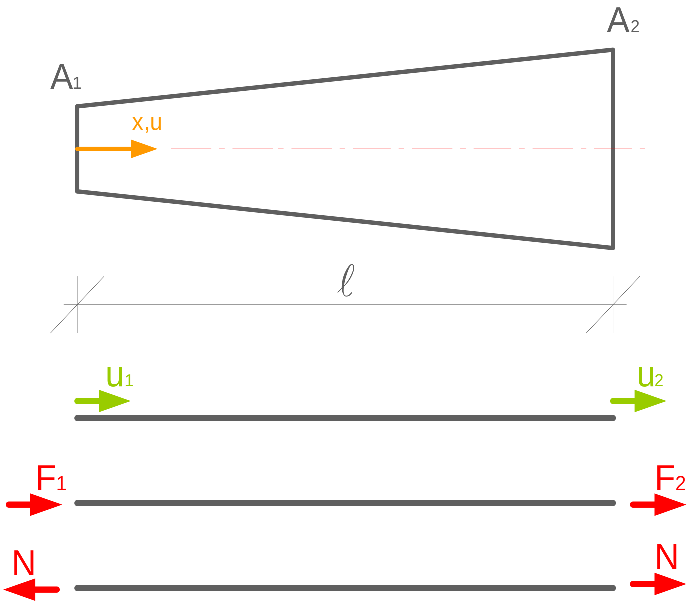
$$F_1 = -N = -\frac{E(A_2 - A_1)}{\ell \cdot \ln(A_2 / A_1)} \cdot (u_2 - u_1) \qquad F_2 = N = \frac{E(A_2 - A_1)}{\ell \cdot \ln(A_2 / A_1)} \cdot (u_2 - u_1)$$

Matrix notation:

$$\frac{E(A_2 - A_1)}{\ell \cdot \ln(A_2 / A_1)} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Example: Analytical solution

Stiffness matrix:



Displacements in the truss element

$$u = \frac{\ln\left(\frac{\ell \cdot A_1 + x(A_2 - A_1)}{\ell \cdot A_1}\right)}{\ln\left(\frac{A_2}{A_1}\right)}(u_2 - u_1) + u_1$$

Stresses in the truss element

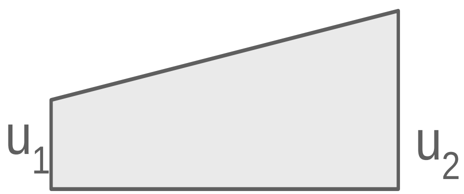
$$\sigma_x = \frac{E \cdot (A_2 - A_1)}{(\ell \cdot A_1 + x(A_2 - A_1)) \cdot \ln(A_2 / A_1)} \cdot (u_2 - u_1)$$

Stiffness matrix

$$\frac{E \cdot (A_2 - A_1)}{\ell \cdot \ln(A_2 / A_1)} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Example: FE solution – linear shape functions

Stiffness matrix:



Displacements:

Assumption of a linear distribution between the nodes.

$$u = u_1 + \frac{x}{\ell} \cdot (u_2 - u_1)$$

Strains:

$$\varepsilon_x = \frac{du}{dx} = \frac{1}{\ell} \cdot (u_2 - u_1)$$

Stresses:

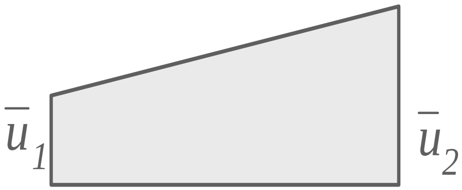
$$\sigma_x = E \cdot \varepsilon_x = \frac{E}{\ell} \cdot (u_2 - u_1)$$

The fulfillment of the equilibrium of the forces (e.g. $F_1 = \sigma_x \cdot A_1, F_2 = \sigma_x \cdot A_2$) is here not possible due to the assumption for the displacements.

Instead, the principle of virtual displacements will be used.

Example: FE solution – linear shape functions

Stiffness matrix:



Virtual displacements:
linear distribution as for real displacements

$$\bar{u} = \bar{u}_1 + \frac{x}{\ell} \cdot \left(\bar{u}_2 - \bar{u}_1\right)$$

Virtual Strains:

$$\bar{\varepsilon}_x = \frac{d\bar{u}}{dx} = \frac{1}{\ell} \cdot \left(\bar{u}_2 - \bar{u}_1\right)$$

Principle of virtual displacements : $\boxed{\bar{W}_i = \bar{W}_a}$

$$\bar{W}_i = \int_0^\ell A_x \cdot \sigma_x \cdot \bar{\varepsilon}_x \, dx$$

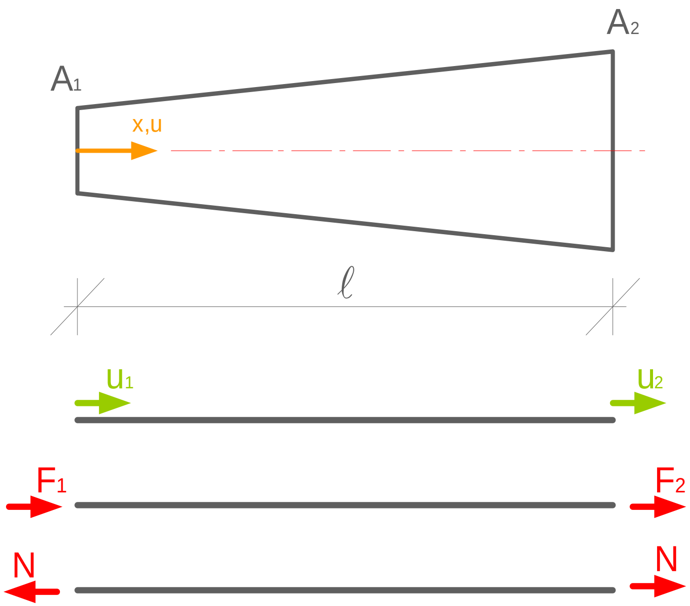
$$= \int_0^\ell \left(A_1 + \frac{x}{\ell}\left(A_2 - A_1\right)\right) \cdot \frac{E}{\ell} \cdot \left(u_2 - u_1\right) \cdot \frac{1}{\ell} \cdot \left(\bar{u}_2 - \bar{u}_1\right) dx$$

$$= \left(A_1 \cdot x + \frac{x^2}{2 \cdot \ell}\left(A_2 - A_1\right)\right) \cdot \frac{E}{\ell} \cdot \left(u_2 - u_1\right) \cdot \frac{1}{\ell} \cdot \left(\bar{u}_2 - \bar{u}_1\right) \Big|_0^\ell = \frac{A_1 + A_2}{2} \cdot \frac{E}{\ell} \cdot \left(u_2 - u_1\right) \cdot \left(\bar{u}_2 - \bar{u}_1\right)$$

W_i = internal virtual work
W_a = external virtual work

Example: FE solution – linear shape functions

Stiffness matrix:



Principle of the virtual displacements: $\overline{W}_i = \overline{W}_a$

$$\overline{W}_i = \frac{A_1 + A_2}{2} \cdot \frac{E}{\ell} \cdot (u_2 - u_1) \cdot (-\overline{u}_1 + \overline{u}_2)$$

$$\overline{W}_a = F_1 \cdot \overline{u}_1 + F_2 \cdot \overline{u}_2$$

a) $\overline{u}_1 = 1; \quad \overline{u}_2 = 0 \quad \Rightarrow \quad \frac{E}{\ell} \cdot \frac{A_1 + A_2}{2} \cdot (u_1 - u_2) = F_1$

b) $\overline{u}_1 = 0; \quad \overline{u}_2 = 1 \quad \Rightarrow \quad \frac{E}{\ell} \cdot \frac{A_1 + A_2}{2} \cdot (-u_1 + u_2) = F_2$

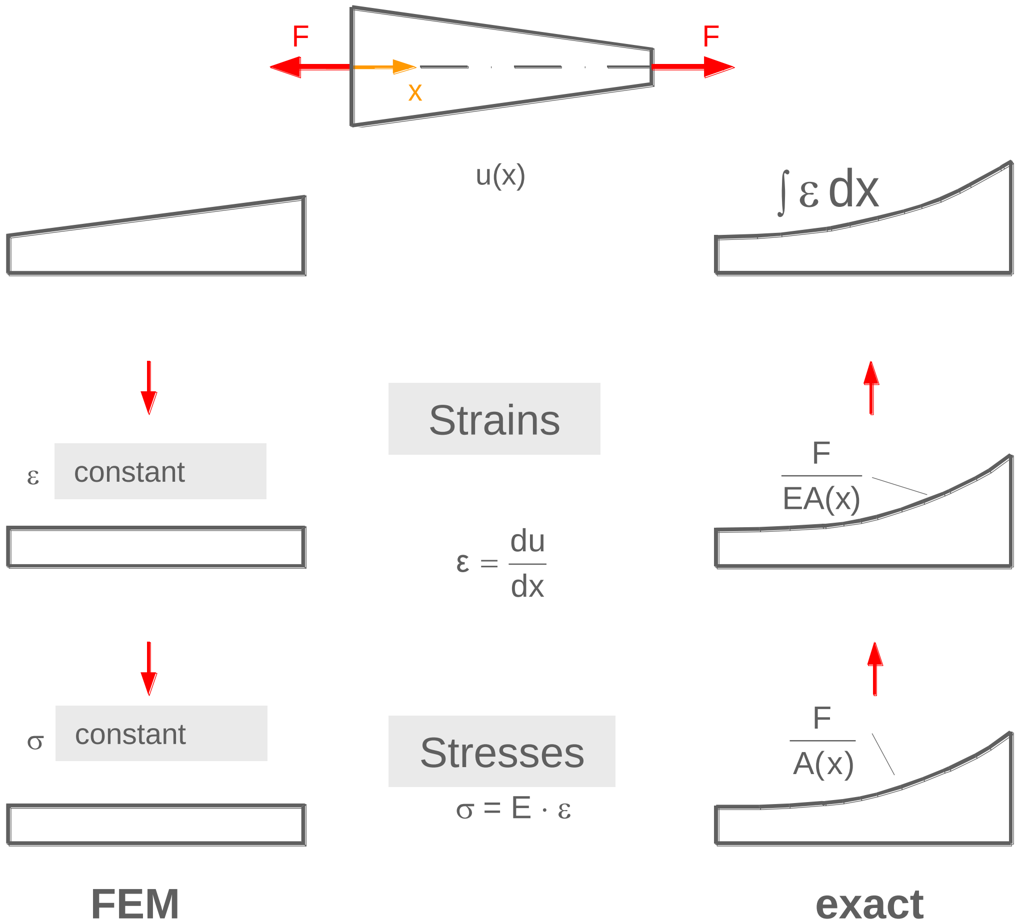
Stiffness matrix

$$\frac{E \cdot (A_1 + A_2)}{2 \cdot \ell} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Example: FE approximation versus exact solution

$$u = u_1 + \frac{x}{\ell} \cdot (u_2 - u_1)$$

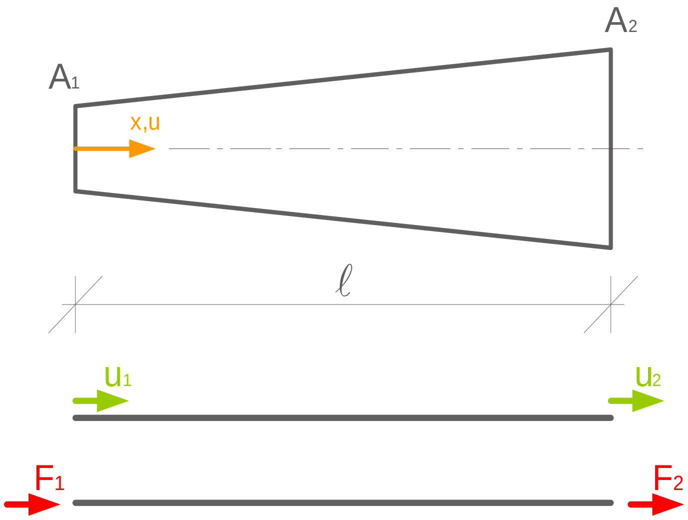
$$\sigma_x = \frac{E}{\ell} \cdot (u_2 - u_1)$$



$$\sigma_x = \frac{N \cdot \ell}{A_1 \cdot \ell + x \cdot (A_2 - A_1)}$$

Example: FE assumption and exact solution

Stiffness matrix



Exact Solution

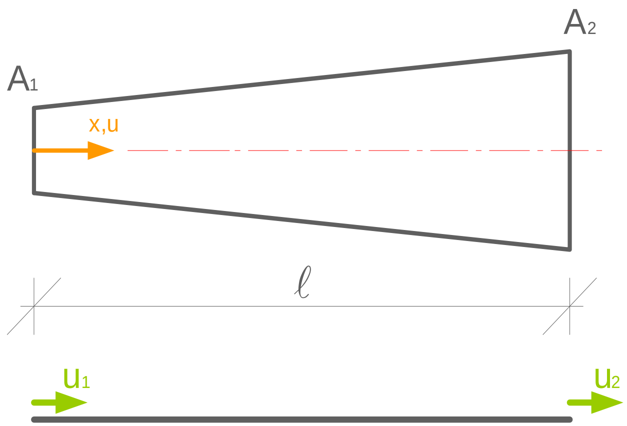
$$\frac{E \cdot (A_2 - A_1)}{\ell \cdot \ln(A_2 / A_1)} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

FEM

$$\frac{E \cdot (A_1 + A_2)}{2 \cdot \ell} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Example: FE solution – linear shape functions

Numerical example: Discretization of a bar into one element



A_1 = 500 cm^2 A_2 = 100 cm^2
E = 1000 kN/cm^2 F = 100 kN

Stiffness relationships:

$$\frac{E}{\ell}\cdot\frac{(A_1+A_2)}{2}\cdot\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\cdot\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}F_1\\ F_2\end{bmatrix}$$ with $u_1 = 0$

Displacement:

$$\frac{E}{\ell}\cdot\frac{(A_1+A_2)}{2}\cdot u_2=F$$

$$\frac{1000}{500}\cdot\frac{500+100}{2}\cdot u_2=100$$

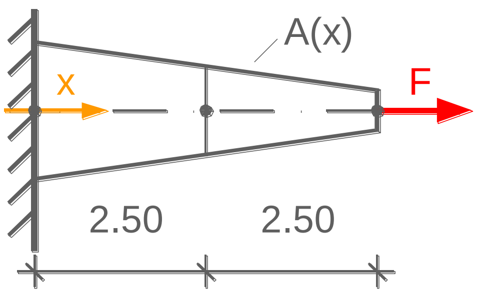
$$u_2=\frac{100}{600}=0.167\,[\mathrm{cm}]$$

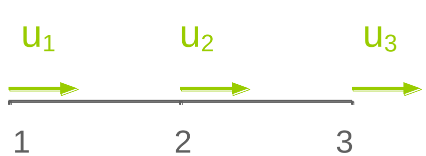
Element stress:

$$\sigma_x=\frac{E}{\ell}\cdot(-u_1+u_2)=\frac{1000}{500}\cdot(0+0.167)=0.333\,[\mathrm{kN/cm}^2]$$

Example: FE solution – linear shape functions

Numerical example: Discretization of a bar into two elements





Cross section area with x = 250 [cm]:

$$A = A_1 + \frac{x}{\ell}\cdot(A_2 - A_1)$$

$$= 500 + \frac{250}{500}\cdot(100 - 500) = 300\ [cm^2]$$

Element stiffness matrices

Element 1: $\frac{E}{\ell}\cdot\frac{(A_1+A_2)}{2}\cdot u_2 = F_2^{(1)}$ $\qquad \frac{1000}{250}\cdot\frac{500+300}{2}\cdot u_2 = F_2^{(1)}$

$$1600\cdot u_2 = F_2^{(1)}$$

Element 2: $\frac{E}{\ell}\cdot\frac{(A_2+A_3)}{2}\cdot\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\cdot\begin{bmatrix}u_2\\ u_3\end{bmatrix} = \begin{bmatrix}F_2^{(2)}\\ F_3^{(2)}\end{bmatrix}$

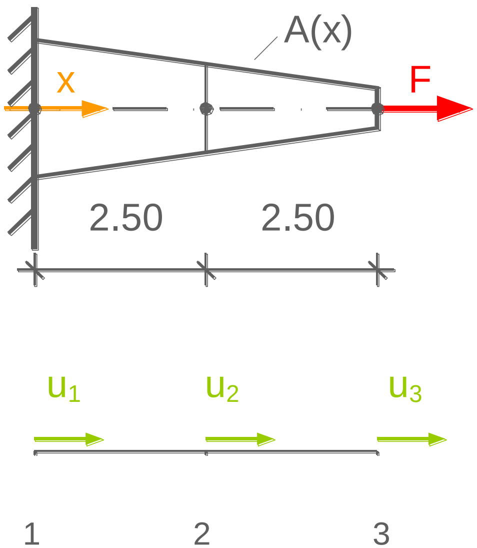
$$\frac{1000}{250}\cdot\frac{300+100}{2}\cdot\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\cdot\begin{bmatrix}u_2\\ u_3\end{bmatrix} = \begin{bmatrix}F_2^{(2)}\\ F_3^{(2)}\end{bmatrix}$$

$$800\cdot\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\cdot\begin{bmatrix}u_2\\ u_3\end{bmatrix} = \begin{bmatrix}F_2^{(2)}\\ F_3^{(2)}\end{bmatrix}$$

Stiffness matrix

Example: FE solution – linear shape functions

Numerical example: Discretization of a bar into two elements



Stiffness matrix

Element 1:

$$1600 \cdot u_2 = F_2^{(1)}$$

Element 2:

$$800 \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_2^{(2)} \\ F_3^{(2)} \end{bmatrix}$$

Global stiffness matrix

$$\begin{bmatrix} 1600+800 & -800 \\ -800 & 800 \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$$

Solution of the system of equations:

u_2 = 0.063 [cm]
u_3 = 0.188 [cm]

$$\sigma_1 = \frac{E}{\ell} \cdot (u_2 - u_1) \qquad \sigma_2 = \frac{E}{\ell} \cdot (u_3 - u_2)$$

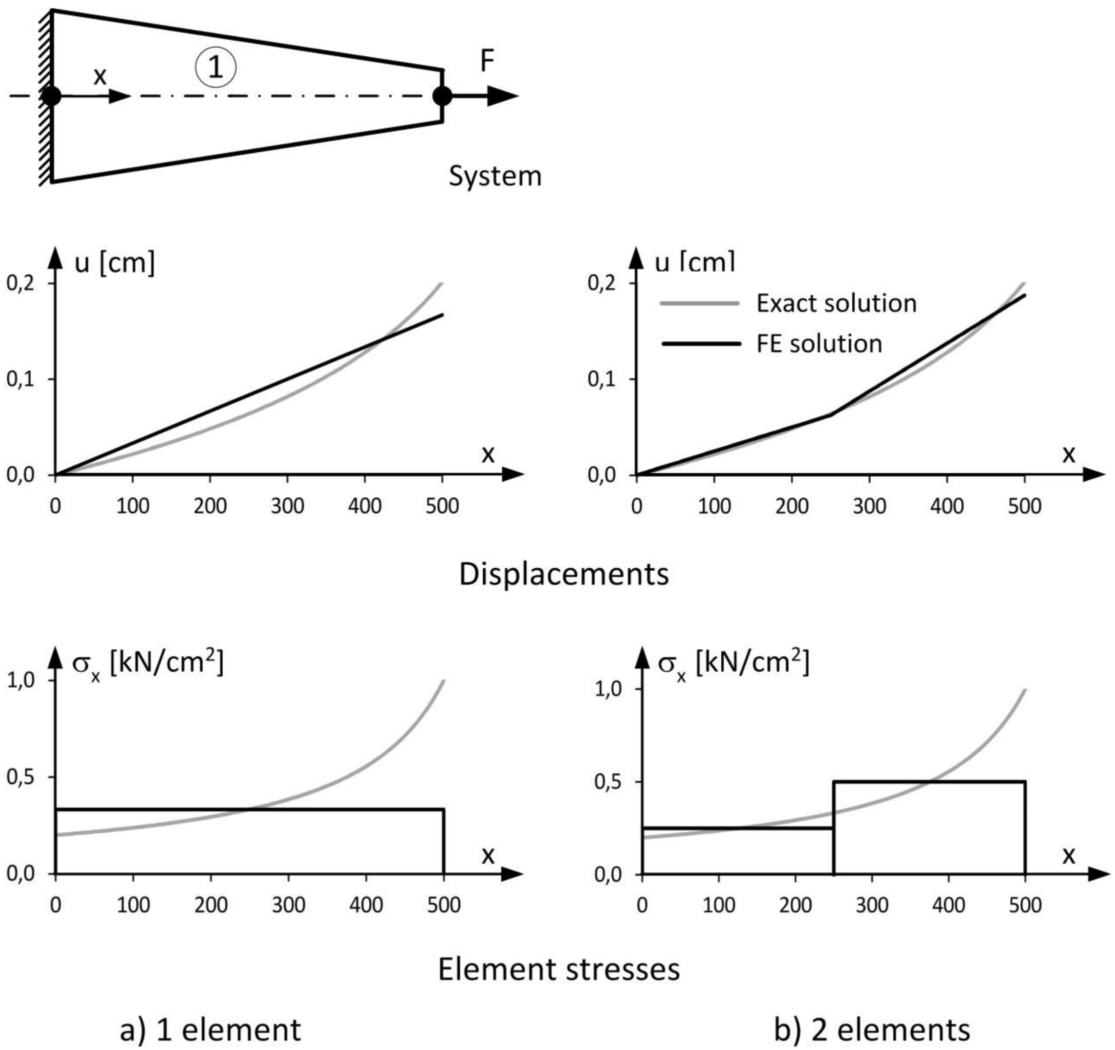
Stresses in the elements 1 and 2:

σ_1 = 0.250 [kN/cm²] σ_2 = 0.500 [kN/cm²]

Example: FE solution and exact solution

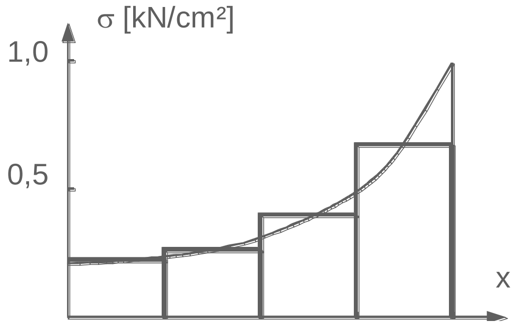
Numerical Example:

FEM approximation with one and two finite elements with linear shape functions



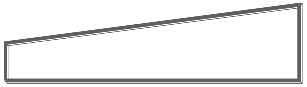
Example: FE solution and exact solution

Accuracy improvement



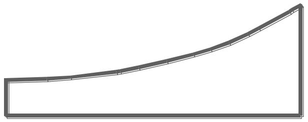
- **Increasing the numbers of elements**

- **Increasing the polynomial degree of the shape functions**



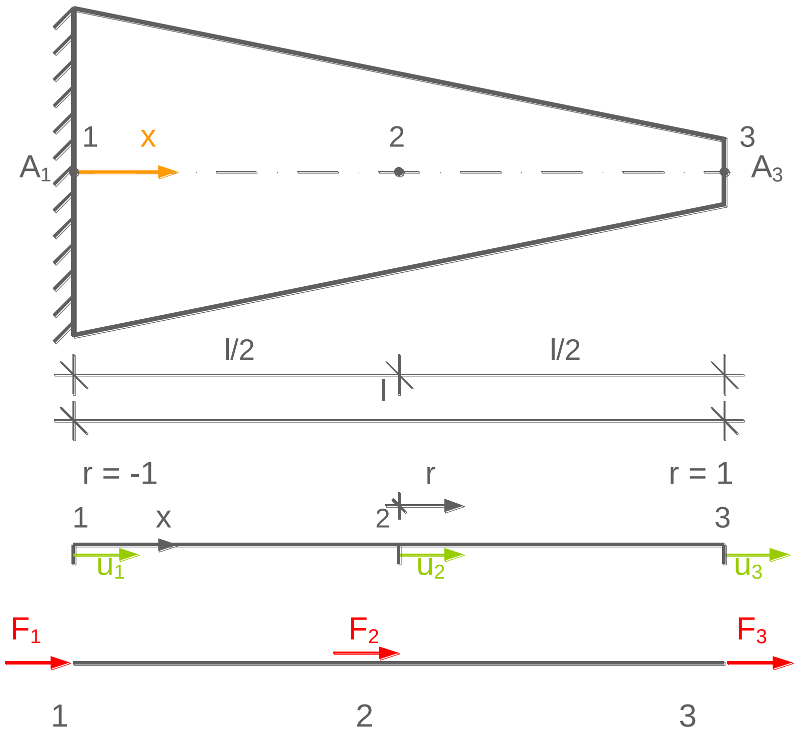
Linear displacement

shape function



Displacement shape functions with polynomial of degree 2, 3, 4, etc.

Example: FE solution – quadratic shape functions



Cross section

$$A = \frac{1}{2}(1-r)A_1 + \frac{1}{2}(1+r)A_3$$

Displacement

$$u = \left[\frac{1}{2}(1-r) - \frac{1}{2}(1-r^2)\right] \cdot u_1 + (1-r^2) \cdot u_2 + \left[\frac{1}{2}(1+r) - \frac{1}{2}(1-r^2)\right] \cdot u_3$$

Strain

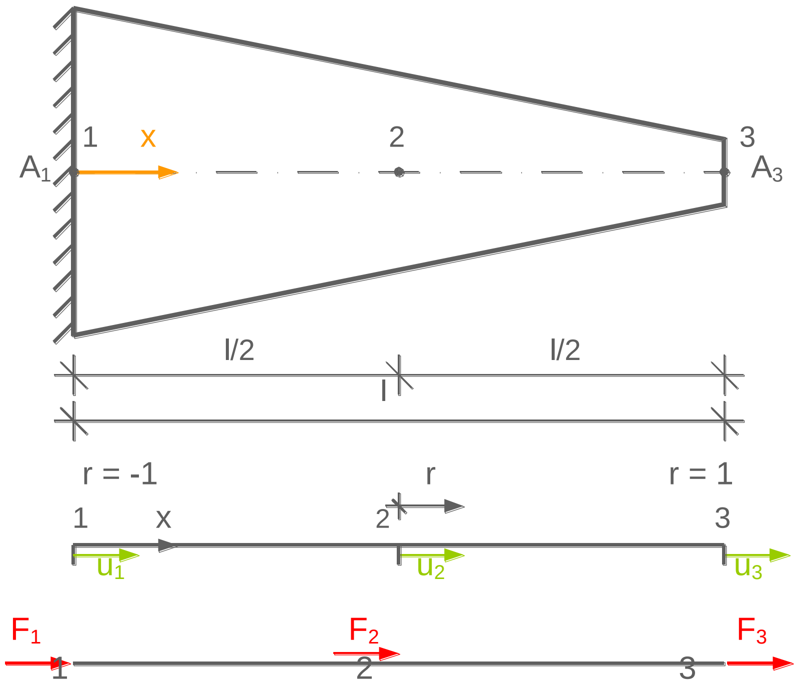
$$\varepsilon = \frac{du}{dx} = \frac{du}{dr} \cdot \frac{dr}{dx}$$

Coordinate **r**

$$r = \frac{2 \cdot x}{\ell} - 1$$

$$\varepsilon = \frac{1}{\ell}(-1+2r) \cdot u_1 - \frac{4}{\ell} r \cdot u_2 + \frac{1}{\ell}(1+2r) \cdot u_3$$

Example: FE solution – quadratic shape functions



Strains

$$\varepsilon = \frac{1}{\ell}(-1+2r)\cdot u_1 - \frac{4}{\ell} r\cdot u_2 + \frac{1}{\ell}(1+2r)\cdot u_3$$

Stress $$\sigma = E\cdot\varepsilon$$

$$\sigma = \frac{E}{\ell}(-1+2r)\cdot u_1 - \frac{4E}{\ell} r\cdot u_2 + \frac{E}{\ell}(1+2r)\cdot u_3$$

Stresses at the nodal points

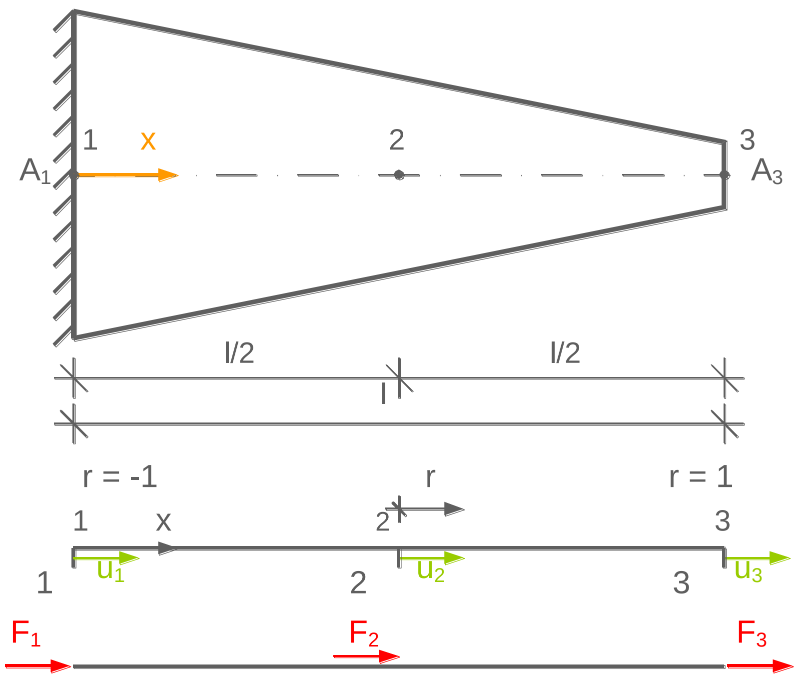
$$\sigma_1 = E\cdot(-3\cdot u_1 + 4\cdot u_2 - u_3)\,/\,l$$

$$\sigma_2 = E\cdot(-u_1 + u_3)\,/\,l$$

$$\sigma_3 = E\cdot(u_1 - 4\cdot u_2 + 3\cdot u_3)\,/\,l$$

Example: FE solution – quadratic shape functions

Principle of virtual displacements



Virtual displacements

$$\bar{u} = \left[\frac{1}{2}(1-r) - \frac{1}{2}(1-r^2)\right]\cdot \bar{u}_1 + (1-r^2)\cdot \bar{u}_2 + \left[\frac{1}{2}(1+r) - \frac{1}{2}(1-r^2)\right]\cdot \bar{u}_3$$

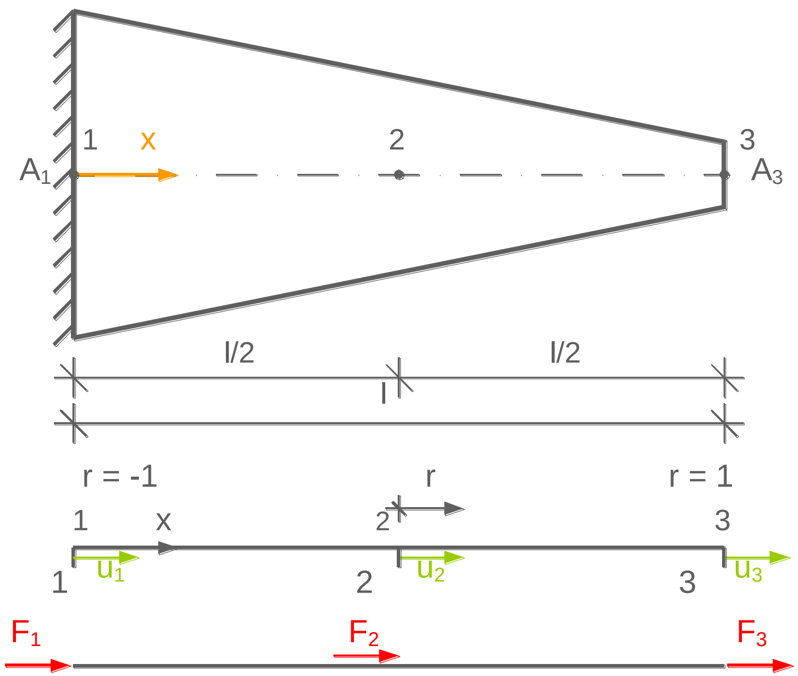
Virtual strains

$$\bar{\varepsilon} = \frac{1}{\ell}(-1+2r)\cdot \bar{u}_1 - \frac{4}{\ell}r\cdot \bar{u}_2 + \frac{1}{\ell}(1+2r)\cdot \bar{u}_3$$

Example: FE solution – quadratic shape functions

Principle of virtual displacements

$$\overline{W}_i = \overline{W}_a$$



$$\int_0^\ell \bar{\varepsilon} \cdot \sigma \cdot A \; dx = F_1 \cdot \bar{u}_1 + F_2 \cdot \bar{u}_2 + F_3 \cdot \bar{u}_3$$

$$\int_0^\ell \left[\frac{1}{\ell}(-1+2r) \cdot \bar{u}_1 - \frac{4}{\ell} r \cdot \bar{u}_2 + \frac{1}{\ell}(1+2r) \cdot \bar{u}_3 \right]$$

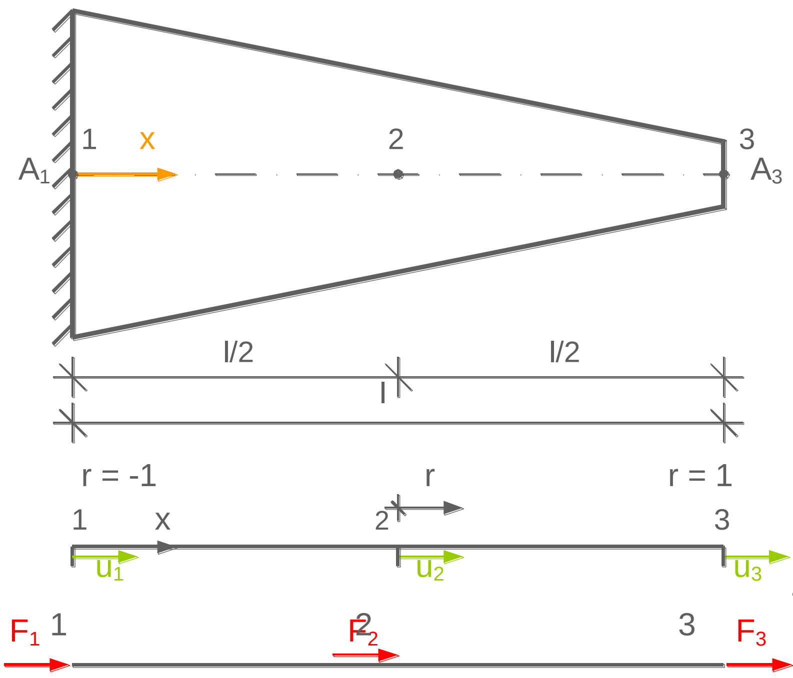
$$\cdot \left[\frac{E}{\ell}(-1+2r) \cdot u_1 - \frac{4E}{\ell} r \cdot u_2 + \frac{E}{\ell}(1+2r) \cdot u_3 \right]$$

$$\cdot \left(A_1 + \frac{x}{\ell}(A_3 - A_1) \right) dx$$

$$= F_1 \cdot \bar{u}_1 + F_2 \cdot \bar{u}_2 + F_3 \cdot \bar{u}_3$$

Example: FE solution – quadratic shape functions

Performing integration



$$\frac{E\,A_1}{\ell}\cdot\left(\left[\left(\frac{7}{3}+\frac{\alpha}{2}\right)\cdot u_1+\left(-\frac{8}{3}-\frac{2}{3}\alpha\right)\cdot u_2+\left(\frac{1}{3}+\frac{\alpha}{6}\right)\cdot u_3\right]\cdot\bar{u}_1\right.$$

$$+\left[\left(-\frac{8}{3}-\frac{2}{3}\alpha\right)\cdot u_1+\left(\frac{16}{3}+\frac{8}{3}\alpha\right)\cdot u_2+\left(-\frac{8}{3}-2\alpha\right)\cdot u_3\right]\cdot\bar{u}_2$$

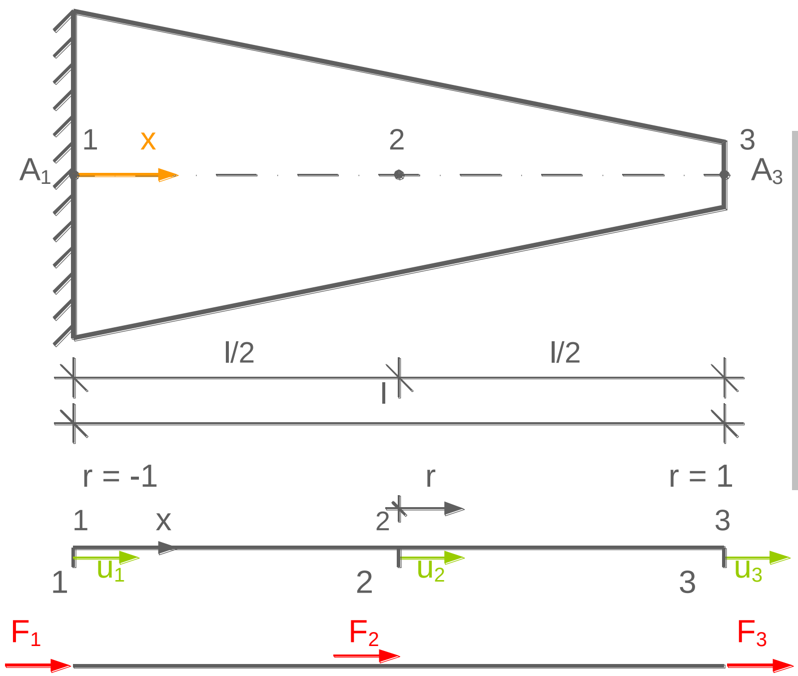
$$\left.+\left[\left(\frac{1}{3}+\frac{\alpha}{6}\right)\cdot u_1+\left(-\frac{8}{3}-2\alpha\right)\cdot u_2+\left(\frac{7}{3}+\frac{11}{6}\alpha\right)\cdot u_3\right]\cdot\bar{u}_3\right)$$

$$=F_1\cdot\bar{u}_1+F_2\cdot\bar{u}_2+F_3\cdot\bar{u}_3$$

with $\alpha=\dfrac{A_3-A_1}{A_1}$

Example: FE solution – quadratic shape functions

Element stiffness matrix



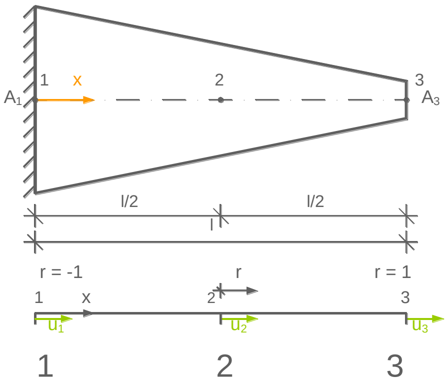
$$\alpha = \frac{A_3 - A_1}{A_1}$$

$$\frac{EA_1}{\ell} \cdot \begin{bmatrix} \frac{7}{3}+\frac{\alpha}{2} & -\frac{8}{3}-\frac{2}{3}\alpha & \frac{1}{3}+\frac{\alpha}{6} \\ -\frac{8}{3}-\frac{2}{3}\alpha & \frac{16}{3}+\frac{8}{3}\alpha & -\frac{8}{3}-2\alpha \\ \frac{1}{3}+\frac{\alpha}{6} & -\frac{8}{3}-2\alpha & \frac{7}{3}+\frac{11}{6}\alpha \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\underline{K}_e \cdot \underline{u}_e = \underline{F}_e$$

Example: FE solution – quadratic shape functions

Numerical example: Discretization of a bar into one element



A1 = 500 [cm²],
A3 = 100 [cm²],
E = 1000 [kN/cm²],
F = 100 [kN]
ℓ = 500 cm

Stiffness relationship:

$$\frac{E A_1}{\ell}\cdot\begin{bmatrix}\frac{16}{3}+\frac{8}{3}\alpha & -\frac{8}{3}-2\alpha\\ -\frac{8}{3}-2\alpha & \frac{7}{3}+\frac{11}{6}\alpha\end{bmatrix}\cdot\begin{bmatrix}u_2\\ u_3\end{bmatrix}=\begin{bmatrix}0\\ F\end{bmatrix}$$

with $u_1=0$

Displacement:

$$1000\cdot\begin{bmatrix}3.200 & -1.067\\ -1.067 & 0.867\end{bmatrix}\cdot\begin{bmatrix}u_2\\ u_3\end{bmatrix}=\begin{bmatrix}0\\ 100\end{bmatrix}$$

$$u_2 = 0.065\ [\mathrm{cm}]$$
$$u_3 = 0.196\ [\mathrm{cm}]$$

Element stresses:

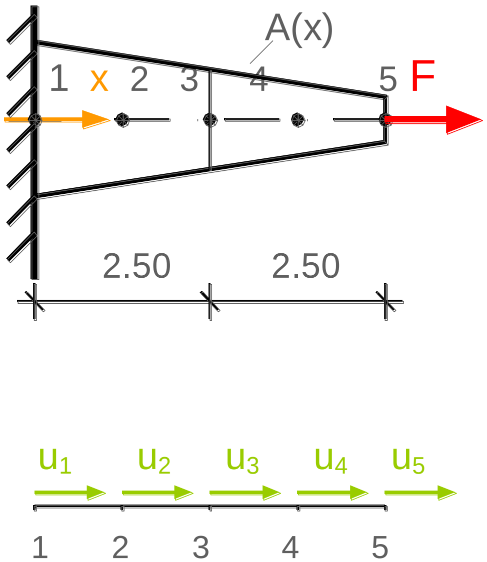
$$\sigma_1 = E\cdot(-3\cdot u_1 + 4\cdot u_2 - u_3)\,/\,l = 1000\cdot(4\cdot 0.065 - 0.196)/500 = 0.128\ [\mathrm{kN/cm^2}]$$

$$\sigma_2 = E\cdot(-u_1 + u_3)\,/\,l = 1000\cdot 0.196/500 = 0.392\ [\mathrm{kN/cm^2}]$$

$$\sigma_3 = E\cdot(u_1 - 4\cdot u_2 + 3\cdot u_3)\,/\,l = 1000\cdot(-4\cdot 0.065 + 3\cdot 0.196)/500 = 0.656\ [\mathrm{kN/cm^2}]$$

Example: FE solution – quadratic shape functions

Numerical example: Discretization of a bar into two elements



$A_1 = 500\ [cm^2]$, $A_3 = 300\ [cm^2]$,
$E = 1000\ [kN/cm^2]$, $\ell = 250\ [cm]$

Element 1:

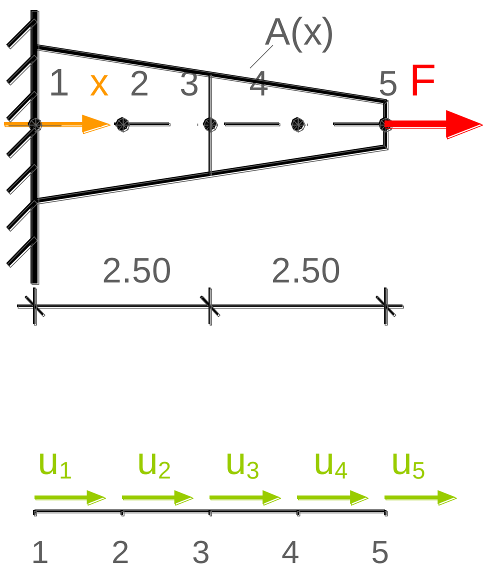
$$\frac{EA_1}{\ell}\cdot\begin{bmatrix}\frac{16}{3}+\frac{8}{3}\alpha & -\frac{8}{3}-2\alpha \\ -\frac{8}{3}-2\alpha & \frac{7}{3}+\frac{11}{6}\alpha\end{bmatrix}\cdot\begin{bmatrix}u_2 \\ u_3\end{bmatrix}=\begin{bmatrix}F_2^{(1)} \\ F_3^{(1)}\end{bmatrix}$$

$$2000\cdot\begin{bmatrix}4.267 & -1.867 \\ -1.867 & 1.600\end{bmatrix}\cdot\begin{bmatrix}u_2 \\ u_3\end{bmatrix}=\begin{bmatrix}F_2^{(1)} \\ F_3^{(1)}\end{bmatrix}$$

Stiffness matrix

Example: FE solution – quadratic shape functions

Numerical example: Diskretization of a bar into two elements



A_1 = 500 [cm²], A_3 = 300 [cm²],
E = 1000 [kN/cm²], ℓ = 250 [cm]

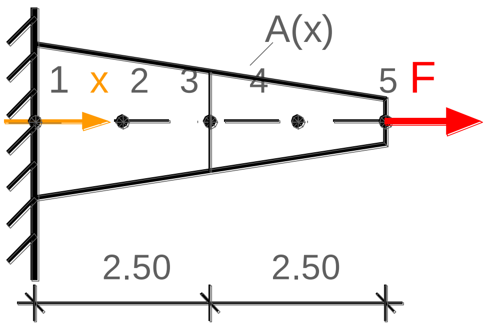
Element 2:

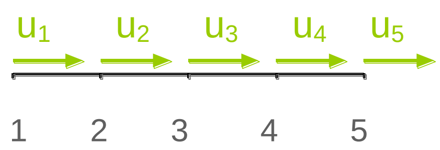
$$\frac{EA_1}{\ell}\cdot\begin{bmatrix} \frac{7}{3}+\frac{\alpha}{2} & -\frac{8}{3}-\frac{2}{3}\alpha & \frac{1}{3}+\frac{\alpha}{6} \\ -\frac{8}{3}-\frac{2}{3}\alpha & \frac{16}{3}+\frac{8}{3}\alpha & -\frac{8}{3}-2\alpha \\ \frac{1}{3}+\frac{\alpha}{6} & -\frac{8}{3}-2\alpha & \frac{7}{3}+\frac{11}{6}\alpha \end{bmatrix}\cdot\begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix}=\begin{bmatrix} F_3^{(2)} \\ F_4^{(2)} \\ F_5^{(2)} \end{bmatrix}$$

$$1200\cdot\begin{bmatrix} 2.000 & -2.222 & 0.222 \\ -2.222 & 3.556 & -1.333 \\ 0.222 & -1.333 & 1.111 \end{bmatrix}\cdot\begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix}=\begin{bmatrix} F_3^{(2)} \\ F_4^{(2)} \\ F_5^{(2)} \end{bmatrix}$$

Stiffness matrix

Example: FE solution – quadratic shape functions





Global Stiffness matrix

$$\begin{bmatrix} 8533 & -3733 & 0 & 0 \\ -3733 & 5600 & -2667 & 267 \\ 0 & -2667 & 4267 & -1600 \\ 0 & 267 & -1600 & 1333 \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100 \end{bmatrix}$$

Solution of the system of equations:

$u_2 = 0.028$ [cm] $\quad u_3 = 0.064$ [cm]

$u_4 = 0.115$ [cm] $\quad u_5 = 0.200$ [cm]

Element stresses:

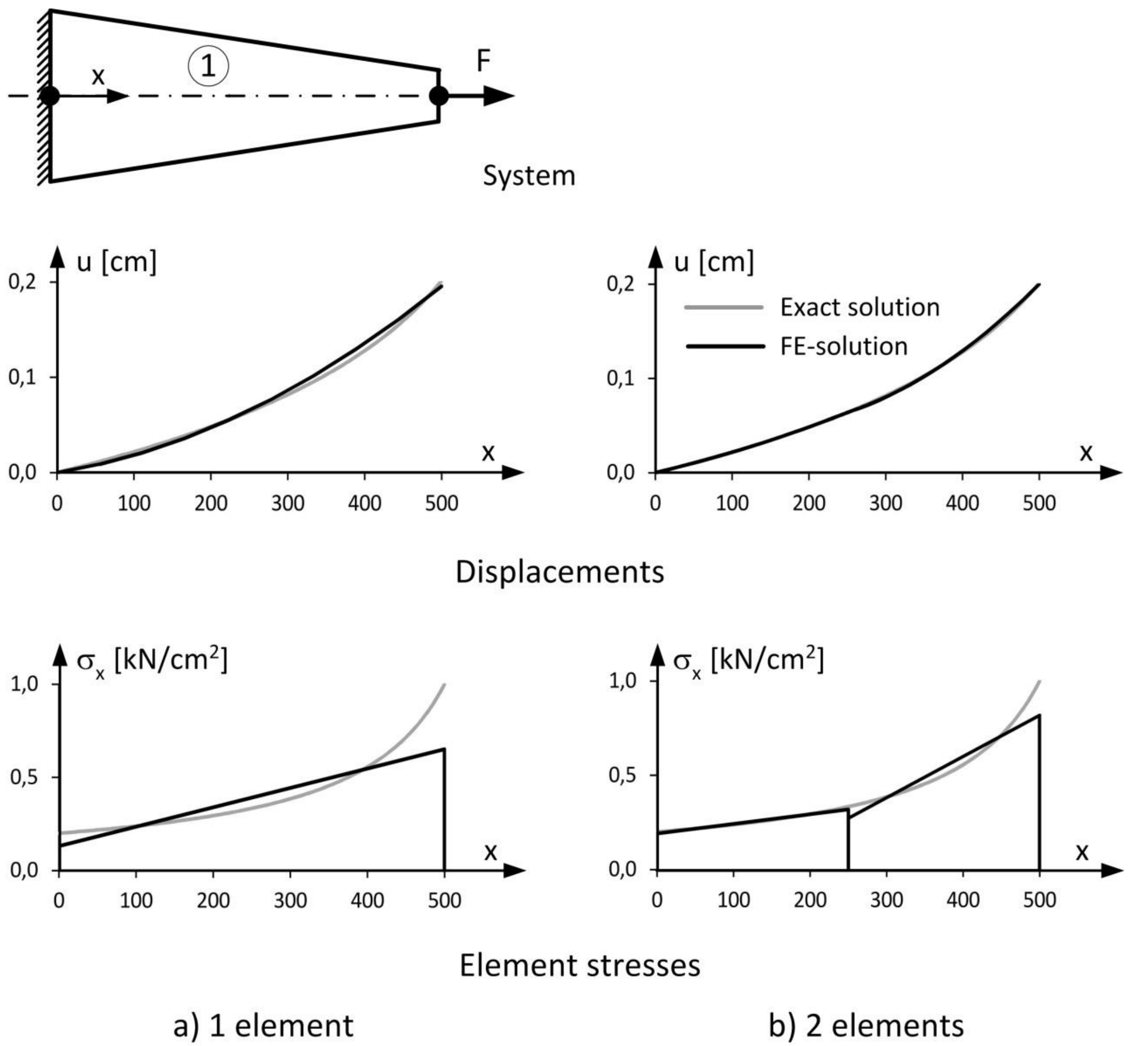
Stresses in element 1: $\sigma_1 = 0.191$ [kN/cm²] $\sigma_2 = 0.255$ [kN/cm²] $\sigma_3 = 0.319$ [kN/cm²]

Stresses in element 2: $\sigma_1 = 0.273$ [kN/cm²] $\sigma_2 = 0.545$ [kN/cm²] $\sigma_3 = 0.818$ [kN/cm²]

Example: FE solution and exact solution

Numerical example:

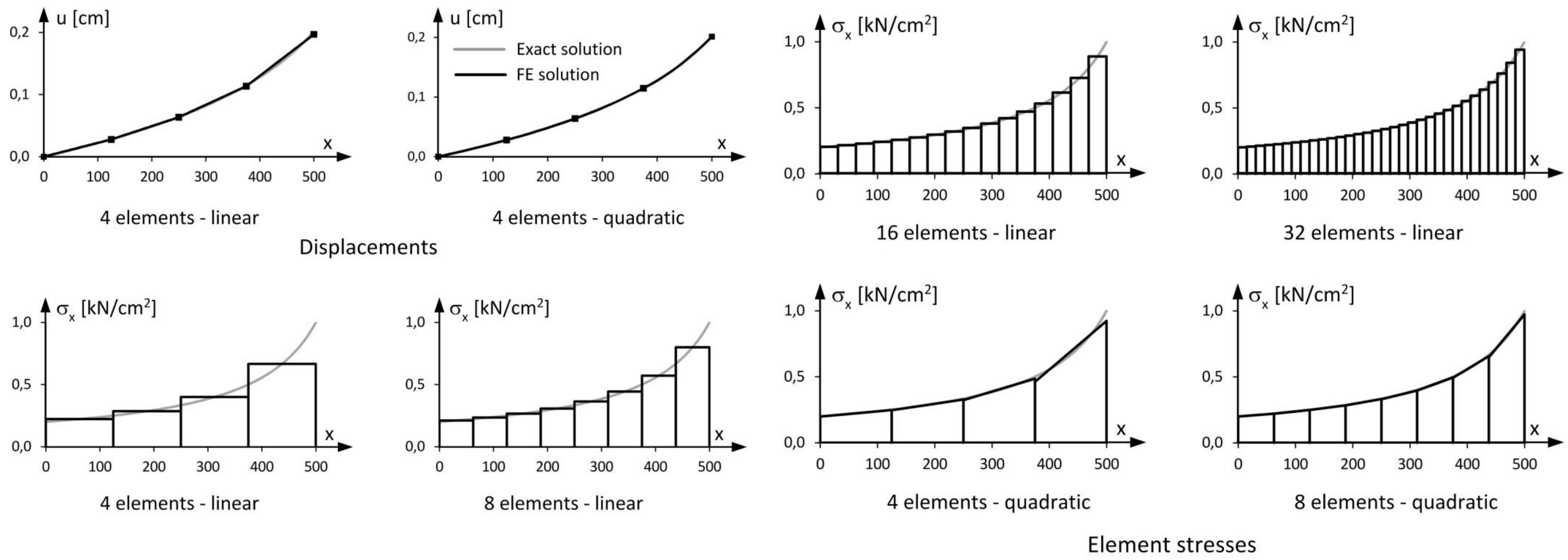
FEM approximation with one and two finite elements with quadratic shape functions



Example: FE solution and exact solution

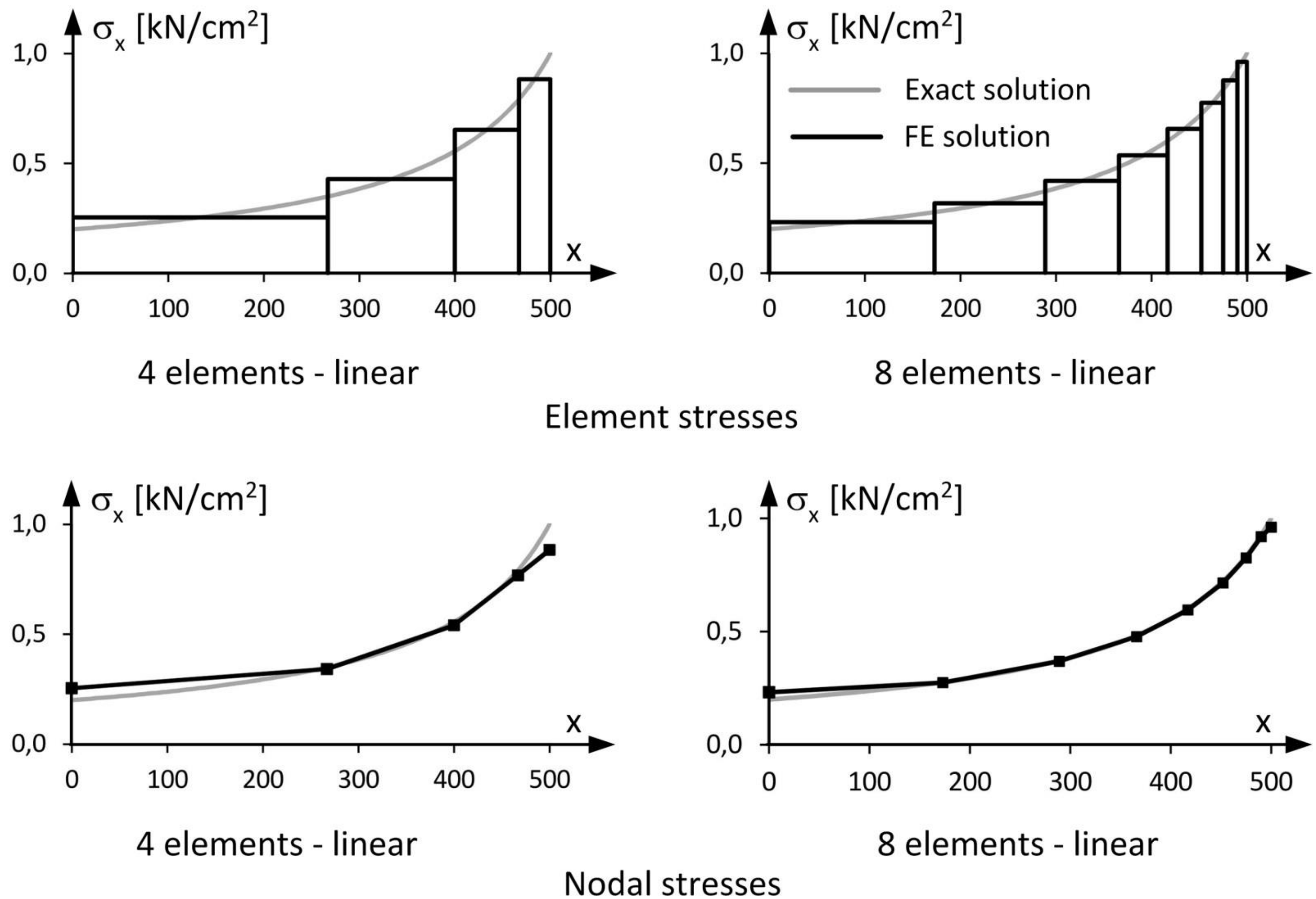
Numerical example:

FEM approximation with 4 - 32 elements with linear and quadratic shape functions



Example: Truss element with linear variable cross section area

Adaption of the element size to the stress gradient



Properties of the Finite element method approximation

a) The finite element solution approximates the exact solution. Its accuracy is increased by an augmentation of the number of elements or a reduction of the element size.

b) Elements with higher order shape functions possess greater accuracy than elements with low shape functions.

c) For elements based on displacement shape functions only, the approximated nodal point displacements are in general too small, i.e. the system behaves too stiffly.

d) The finite element approximation is better in regions with low stress gradient, compared to regions with higher stress gradient, if the element size is uniform.

e) The element stresses in the middle of the element have a greater accuracy than those at the element boundaries.

f) The ’jump’ of the stresses between two adjacent elements is a measure for the accuracy of the analysis at this point.

End

Introduction

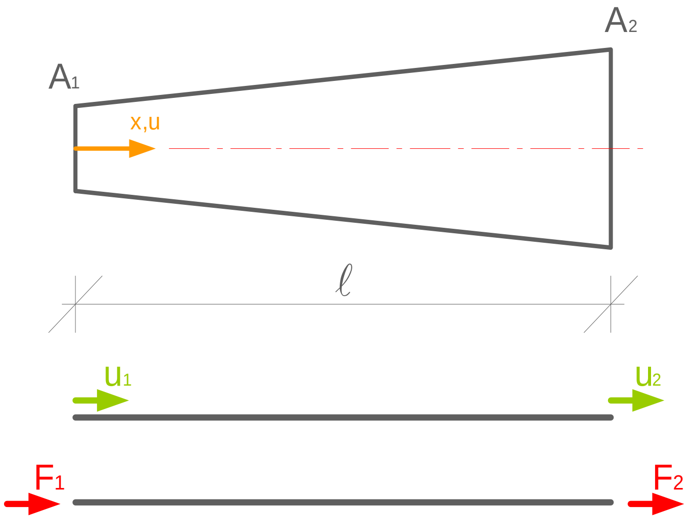
Truss and beam structures

3 Plate and shell structures

Modeling

Example: FE assumption and exact solution

Stiffness matrix



Exact Solution

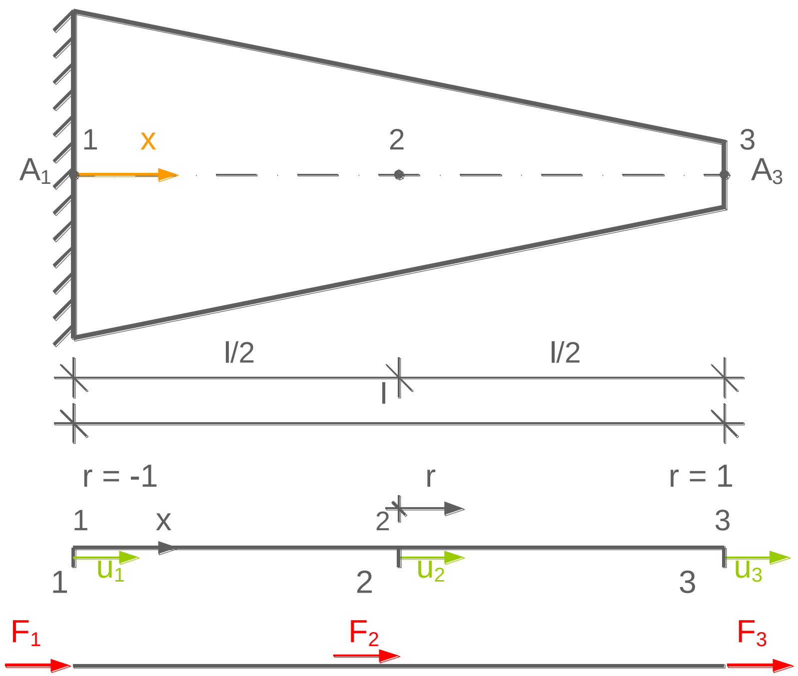
$$\frac{E\cdot(A_2-A_1)}{\ell\cdot\ln(A_2/A_1)}\cdot\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\cdot\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}F_1\\ F_2\end{bmatrix}$$

FEM

$$\frac{E\cdot(A_1+A_2)}{2\cdot\ell}\cdot\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\cdot\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}F_1\\ F_2\end{bmatrix}$$

Example: FE solution – quadratic shape functions

Stiffness matrix:



$$\frac{EA_1}{\ell}\cdot\begin{bmatrix} \frac{7}{3}+\frac{\alpha}{2} & -\frac{8}{3}-\frac{2}{3}\alpha & \frac{1}{3}+\frac{\alpha}{6} \\ -\frac{8}{3}-\frac{2}{3}\alpha & \frac{16}{3}+\frac{8}{3}\alpha & -\frac{8}{3}-2\alpha \\ \frac{1}{3}+\frac{\alpha}{6} & -\frac{8}{3}-2\alpha & \frac{7}{3}+\frac{11}{6}\alpha \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}=\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\underline{\mathbf{K}}_e \cdot \underline{\mathbf{u}}_e = \underline{\mathbf{F}}_e$$

with $\alpha=\dfrac{A_3-A_1}{A_1}$