Finite Elements in Structural Analysis

Introduction

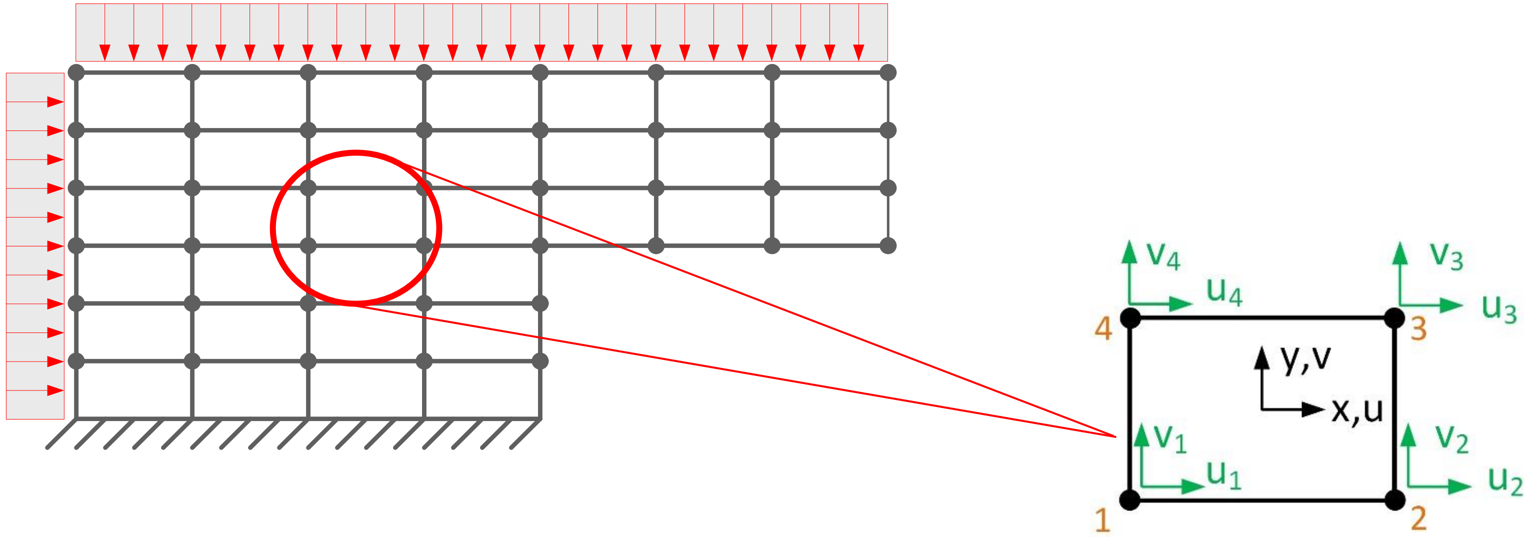
Truss and beam structures

Plate and shell structures

Modeling

Rectangular plane stress element

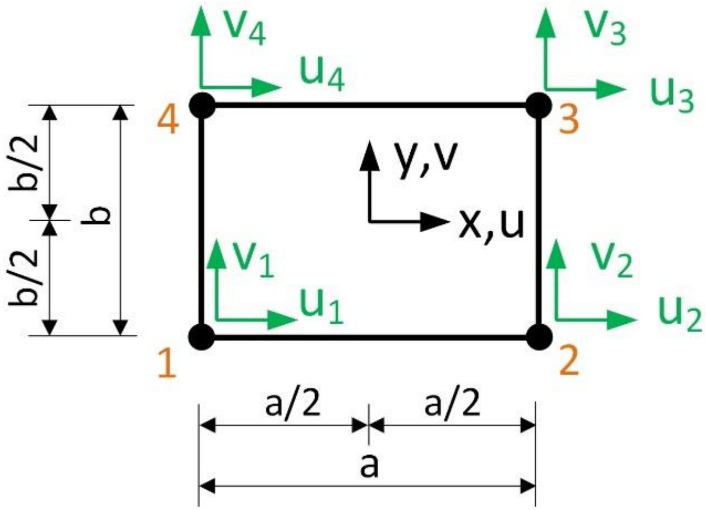
Discretization of a plate into finite elements



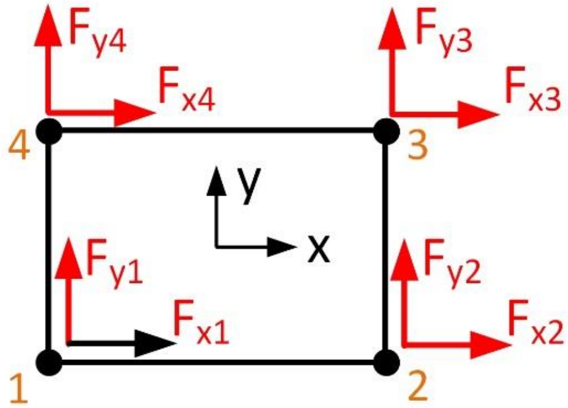
Finite element

Rectangular plane stress element

Degrees of freedom and element forces



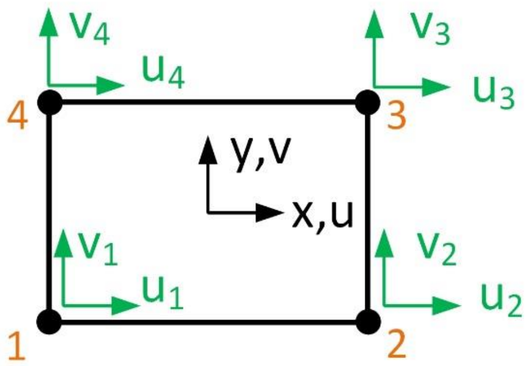
Displacements

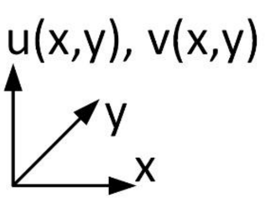


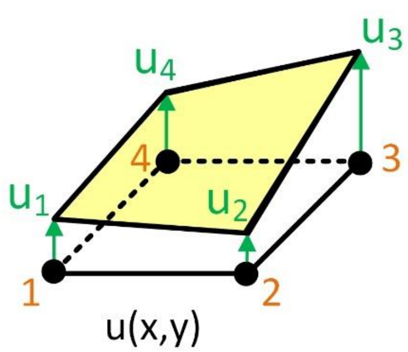
Element forces

Rectangular plane stress element

Shape functions of the displacements







Shape functions of **u**

Bilinear shape function for the displacements:

$$u = \alpha_1 + \alpha_2 \cdot x + \alpha_3 \cdot y + \alpha_4 \cdot x \cdot y$$

$$v = ß_1 + ß_2 \cdot x + ß_3 \cdot y + ß_4 \cdot x \cdot y$$

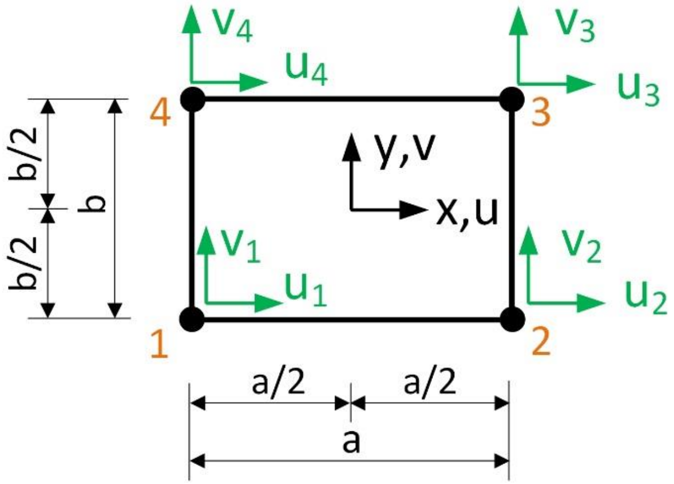
bilinear term

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

$$\underline{u} = \underline{N}_a \cdot \underline{a}$$

Rectangular plane stress element

Shape functions of the displacements



Nodal point displacements

nodal point 1:

$$u_1 = \alpha_1 + \alpha_2\cdot(-a/2) + \alpha_3\cdot(-b/2) + \alpha_4\cdot(-a/2)\cdot(-b/2),$$
$$v_1 = \beta_1 + \beta_2\cdot(-a/2) + \beta_3\cdot(-b/2) + \beta_4\cdot(-a/2)\cdot(-b/2),$$

nodal point 2:

$$u_2 = \alpha_1 + \alpha_2\cdot(a/2) + \alpha_3\cdot(-b/2) + \alpha_4\cdot(a/2)\cdot(-b/2),$$
$$v_2 = \beta_1 + \beta_2\cdot(a/2) + \beta_3\cdot(-b/2) + \beta_4\cdot(a/2)\cdot(-b/2),$$

nodal point 3:

$$u_3 = \alpha_1 + \alpha_2\cdot(a/2) + \alpha_3\cdot(b/2) + \alpha_4\cdot(a/2)\cdot(b/2),$$
$$v_3 = \beta_1 + \beta_2\cdot(a/2) + \beta_3\cdot(b/2) + \beta_4\cdot(a/2)\cdot(b/2),$$

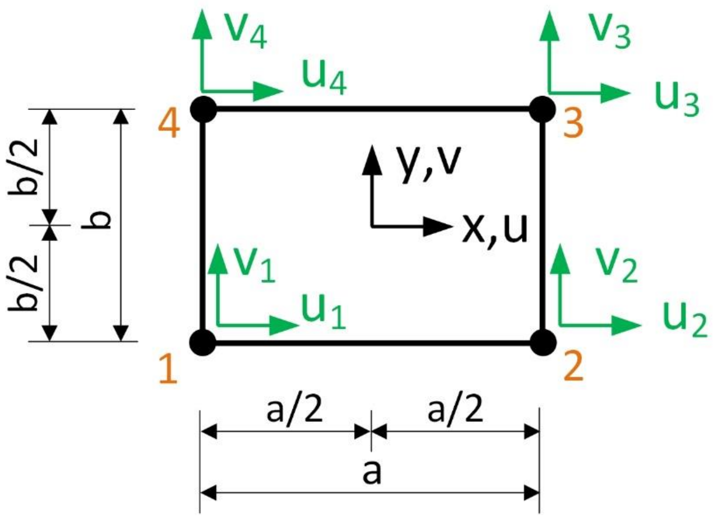
nodal point 4:

$$u_4 = \alpha_1 + \alpha_2\cdot(-a/2) + \alpha_3\cdot(b/2) + \alpha_4\cdot(-a/2)\cdot(b/2)$$
$$v_4 = \beta_1 + \beta_2\cdot(-a/2) + \beta_3\cdot(b/2) + \beta_4\cdot(-a/2)\cdot(b/2).$$

The parameters α_1-α_4 and β_1-β_4 are expressed by the nodal point displacements u_1, v_1 *to* u_4, v_4.

Rectangular element for plates in plane stress

Shape functions of the displacements

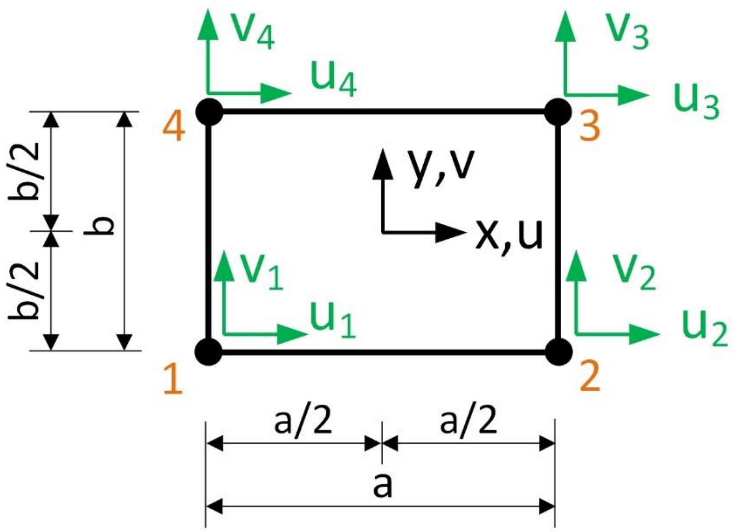


$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{a} & 0 & \frac{1}{a} & 0 & -\frac{1}{a} & 0 \\ -\frac{1}{b} & 0 & -\frac{1}{b} & 0 & \frac{1}{b} & 0 & \frac{1}{b} & 0 \\ \frac{2}{a \cdot b} & 0 & -\frac{2}{a \cdot b} & 0 & \frac{2}{a \cdot b} & 0 & -\frac{2}{a \cdot b} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{a} & 0 & \frac{1}{a} & 0 & \frac{1}{a} & 0 & -\frac{1}{a} \\ 0 & -\frac{1}{b} & 0 & -\frac{1}{b} & 0 & \frac{1}{b} & 0 & \frac{1}{b} \\ 0 & \frac{2}{a \cdot b} & 0 & -\frac{2}{a \cdot b} & 0 & \frac{2}{a \cdot b} & 0 & -\frac{2}{a \cdot b} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$\underline{a} = \underline{A} \cdot \underline{u}_e$$

Rectangular plane stress element

Shape functions of the displacements



Displacements **u**, expressed by **a**

$$\underline{u} = \underline{N}_a \cdot \underline{a}$$

$$\underline{a} = \underline{A} \cdot \underline{u}_e$$

a, expressed by the nodal point displacements **u_e**

$$\underline{u} = \underline{N}_a \cdot \underline{A} \cdot \underline{u}_e$$

or

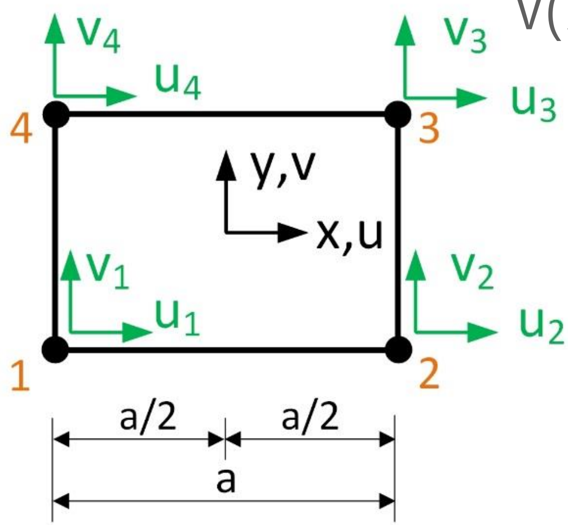
$$\underline{u} = \underline{N} \cdot \underline{u}_e$$

Rectangular plane stress element

Shape functions of the displacements

$$u(x,y) = N_1(x,y)\cdot u_1 + N_2(x,y)\cdot u_2 + N_3(x,y)\cdot u_3 + N_4(x,y)\cdot u_4$$

$$v(x,y) = N_1(x,y)\cdot v_1 + N_2(x,y)\cdot v_2 + N_3(x,y)\cdot v_3 + N_4(x,y)\, v_4$$



$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$\underline{u} = \underline{N} \cdot \underline{u}_e$$

with:

$$N_1 = \frac{1}{4} - \frac{1}{2a}x - \frac{1}{2b}y + \frac{1}{ab}xy \qquad N_2 = \frac{1}{4} + \frac{1}{2a}x - \frac{1}{2b}y - \frac{1}{ab}xy \qquad N_3 = \frac{1}{4} + \frac{1}{2a}x + \frac{1}{2b}y + \frac{1}{ab}xy \qquad N_4 = \frac{1}{4} - \frac{1}{2a}x + \frac{1}{2b}y - \frac{1}{ab}xy$$

Rectangular plane stress element

Shape functions of the displacements

Shape functions

$$N_1=\frac{1}{4}-\frac{1}{2a}x-\frac{1}{2b}y+\frac{1}{ab}xy$$

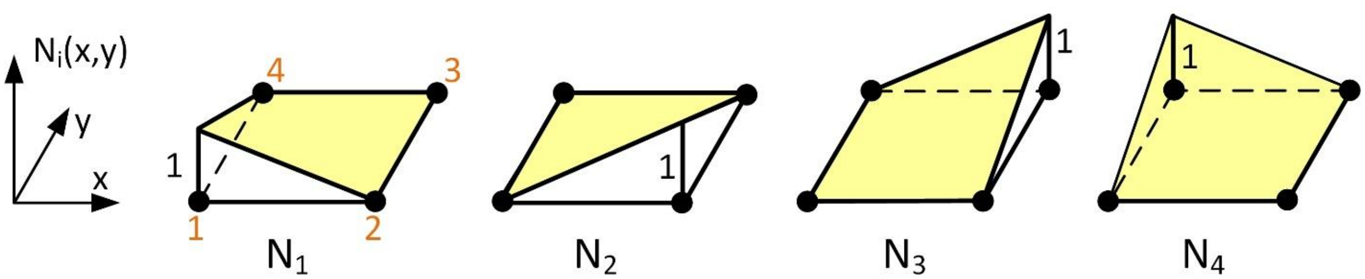
$$N_2=\frac{1}{4}+\frac{1}{2a}x-\frac{1}{2b}y-\frac{1}{ab}xy$$

$$N_3=\frac{1}{4}+\frac{1}{2a}x+\frac{1}{2b}y+\frac{1}{ab}xy$$

$$N_4=\frac{1}{4}-\frac{1}{2a}x+\frac{1}{2b}y-\frac{1}{ab}xy$$

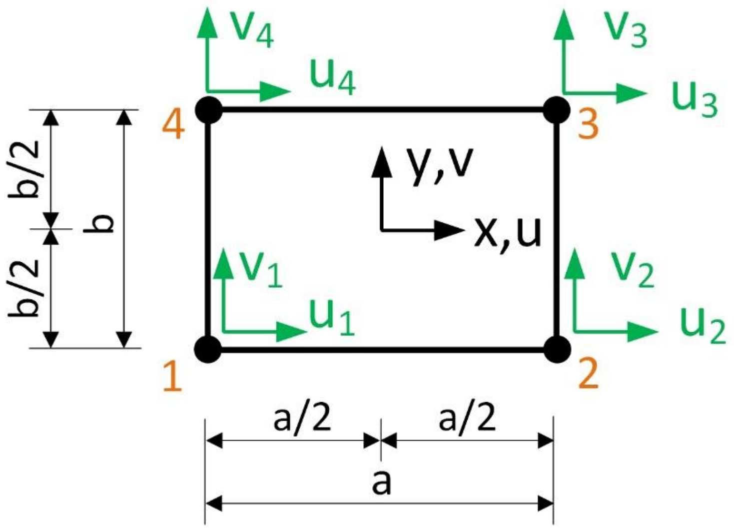
$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$\underline{u}=\underline{N}\cdot\underline{u}_e$$



Rectangular plane stress element

Element displacements



$$\underline{u} = \underline{N} \cdot \underline{u}_e$$

$$u(x,y) = \frac{1}{4}(u_1 + u_2 + u_3 + u_4) + x \cdot \frac{1}{2a}(-u_1 + u_2 + u_3 - u_4) + y\frac{1}{2b}(-u_1 - u_2 + u_3 + u_4) + x \cdot y \cdot \frac{1}{ab}(u_1 - u_2 + u_3 - u_4)$$

$$v(x,y) = \frac{1}{4}(v_1 + v_2 + v_3 + v_4) + x \cdot \frac{1}{2a}(-v_1 + v_2 + v_3 - v_4) + y\frac{1}{2b}(-v_1 - v_2 + v_3 + v_4) + x \cdot y \cdot \frac{1}{ab}(v_1 - v_2 + v_3 - v_4)$$

Rectangular plane stress element

Strains

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}=\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\end{bmatrix}=\begin{bmatrix}\frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0\\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x}\end{bmatrix}\cdot\begin{bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\end{bmatrix}$$

$$\underline{\varepsilon} = \underline{B} \cdot \underline{u}_e$$

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}=\frac{1}{2ab}\begin{bmatrix}2y-b & 0 & -2y+b & 0 & 2y+b & 0 & -2y-b & 0\\ 0 & 2x-a & 0 & -2x-a & 0 & 2x+a & 0 & -2x+a\\ 2x-a & 2y-b & -2x-a & -2y+b & 2x+a & 2y+b & -2x+a & -2y-b\end{bmatrix}\cdot\begin{bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\end{bmatrix}$$

Shape functions

Rectangular plane stress element

Stresses

Strain vector $\underline{\varepsilon} = \underline{B} \cdot \underline{u}_e$

Hooke's law $\underline{\sigma} = \underline{D} \cdot \underline{\varepsilon}$

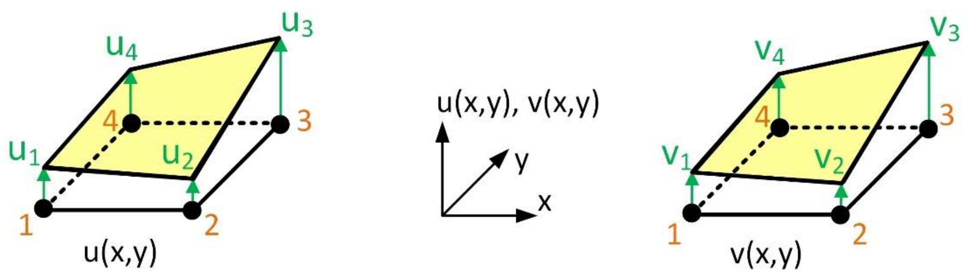
Stress vector $\underline{\sigma} = \underline{D} \cdot \underline{B} \cdot \underline{u}_e$

with
$$\underline{D} = \frac{E}{1-\mu^2} \cdot \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

Rectangular plane stress element

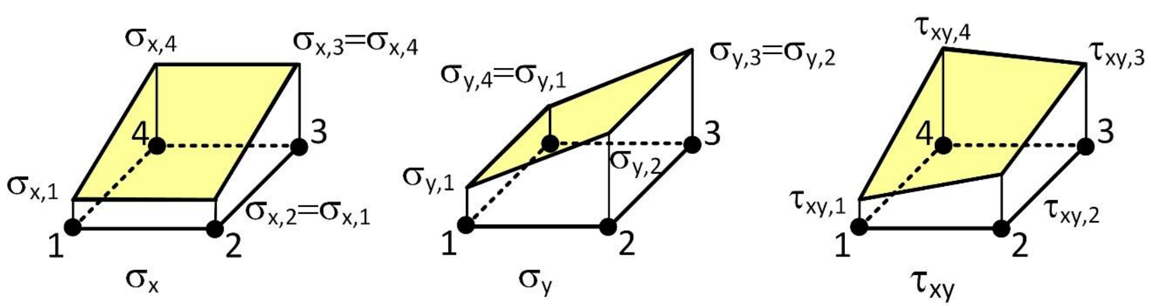
Shape functions of the rectangular plate element and stresses derived thereof

Shape function



bilinear functions

Stresses derived from the shape functions

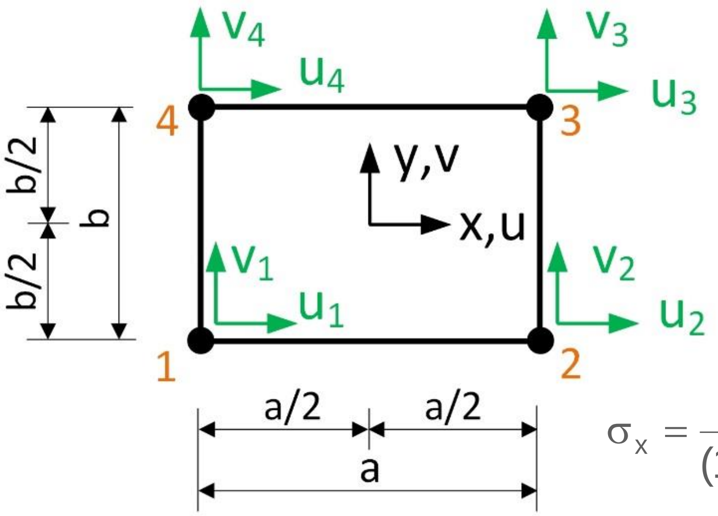


for $\mu = 0$:

σ_x constant in x-direction
linear in y-direction

σ_y constant in y-direction
linear in x-direction

Rectangular plane stress element



Element stresses

$$\underline{\sigma} = \underline{D} \cdot \underline{B} \cdot \underline{u}_e$$

$$\sigma_x = \frac{E}{(1-\mu^2)\cdot 2\cdot a\cdot b}\cdot[(2\cdot y-b)\cdot(u_1-u_2)+(2\cdot y+b)\cdot(u_3-u_4)+ \mu\cdot((2\cdot x-a)\cdot(v_1-v_4)+(2\cdot x+a)\cdot(-v_2+v_3))]$$

$$\sigma_y = \frac{E}{(1-\mu^2)\cdot 2\cdot a\cdot b}\cdot[\mu\cdot((2\cdot y-b)\cdot(u_1-u_2)+(2\cdot y+b)\cdot(u_3-u_4))+ (2\cdot x-a)\cdot(v_1-v_4)+(2\cdot x+a)\cdot(-v_2+v_3)]$$

$$\tau_{xy} = \frac{E}{4\cdot(1+\mu)\cdot a\cdot b}\cdot[(2\cdot y-b)\cdot(v_1-v_2)+(2\cdot y+b)\cdot(v_3-v_4)+ (2\cdot x-a)\cdot(u_1-u_4)+(2\cdot x+a)\cdot(u_3-u_2)]$$

Rectangular plane stress element

Principle of virtual displacements

Principle of work:

$$\overline{W}_i = \overline{W}_a$$

Internal work:

$$\overline{W}_i = t \int \underline{\overline{\varepsilon}}^T \cdot \underline{\sigma} \; dx \; dy$$

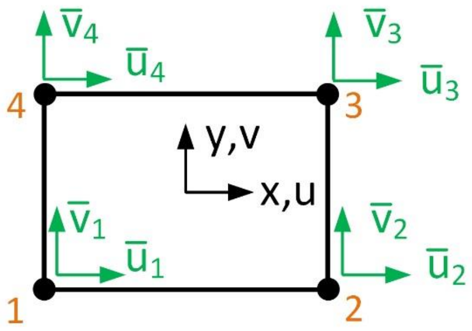
$$\underline{\overline{\varepsilon}} = \underline{B} \cdot \underline{\overline{u}}_e \qquad \underline{\overline{\varepsilon}}^T = \underline{\overline{u}}_e^T \cdot \underline{B}^T \qquad \underline{\sigma} = \underline{D} \cdot \underline{\varepsilon} = \underline{D} \cdot \underline{B} \cdot \underline{u}_e$$

$$\overline{W}_i = t \cdot \int \underline{\overline{u}}_e^T \cdot \underline{B}^T \cdot \underline{D} \cdot \underline{B} \cdot \underline{u}_e \; dx \; dy$$

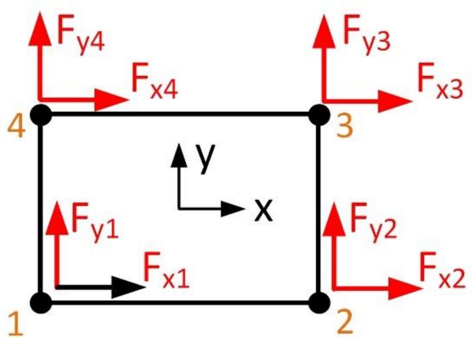
$$\overline{W}_i = \underline{\overline{u}}_e^T \cdot t \int \underline{B}^T \cdot \underline{D} \cdot \underline{B} \; dx \; dy \cdot \underline{u}_e$$

Rectangular plane stress element

Principle of virtual displacements



Virtual displacements



Real forces

External work

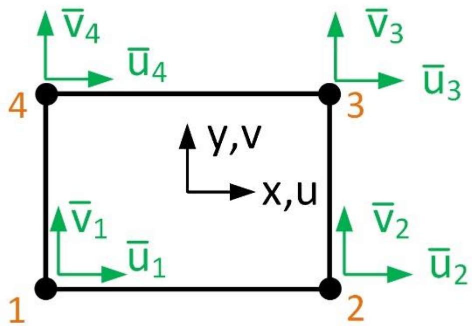
done by the element nodal forces:

$$\bar{W}_a = \underline{\bar{u}}_e^T \cdot \underline{F}_e$$

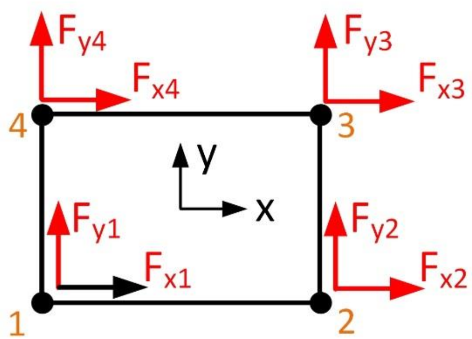
$$\bar{W}_a = \begin{bmatrix} \bar{u}_1 & \bar{v}_1 & \bar{u}_2 & \bar{v}_2 & \bar{u}_3 & \bar{v}_3 & \bar{u}_4 & \bar{v}_4 \end{bmatrix} \cdot \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}$$

Rectangular plane stress element

Principle of virtual displacements



Virtual displacements



Real forces

$$\overline{W}_a = \underline{\bar{u}}_e^T \cdot \underline{F}_e \qquad \overline{W}_i = \underline{\bar{u}}_e^T \cdot t \int \underline{B}^T \cdot \underline{D} \cdot \underline{B} \; dx \; dy \cdot \underline{u}_e$$

$$\overline{W}_i = \overline{W}_a$$

$$\underline{\bar{u}}_e^T \cdot t \int \underline{B}^T \cdot \underline{D} \cdot \underline{B} \; dx \; dy \cdot \underline{u}_e = \underline{\bar{u}}_e^T \cdot \underline{F}_e$$

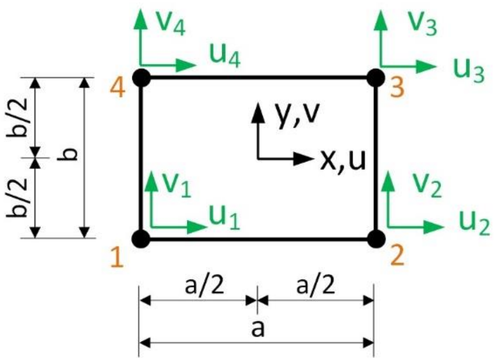
This applies to all virtual displacements $\underline{\bar{u}}_e$

$$\rightarrow \quad t \cdot \int \underline{B}^T \cdot \underline{D} \cdot \underline{B} \; dx \; dy \cdot \underline{u}_e = \underline{F}_e$$

$$\underline{K}^{(e)} \cdot \underline{u}_e = \underline{F}_e \qquad \underline{K}^{(e)} = t \cdot \int \underline{B}^T \cdot \underline{D} \cdot \underline{B} \; dx \; dy$$

Rectangular plane stress element

Stiffness matrix of a rectangular plate element



$$\frac{E\cdot t}{12\cdot(1-\mu^2)}\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} & k_{57} & k_{58} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} & k_{67} & k_{68} \\ k_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} & k_{78} \\ k_{81} & k_{82} & k_{83} & k_{84} & k_{85} & k_{86} & k_{87} & k_{88} \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}=\begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}$$

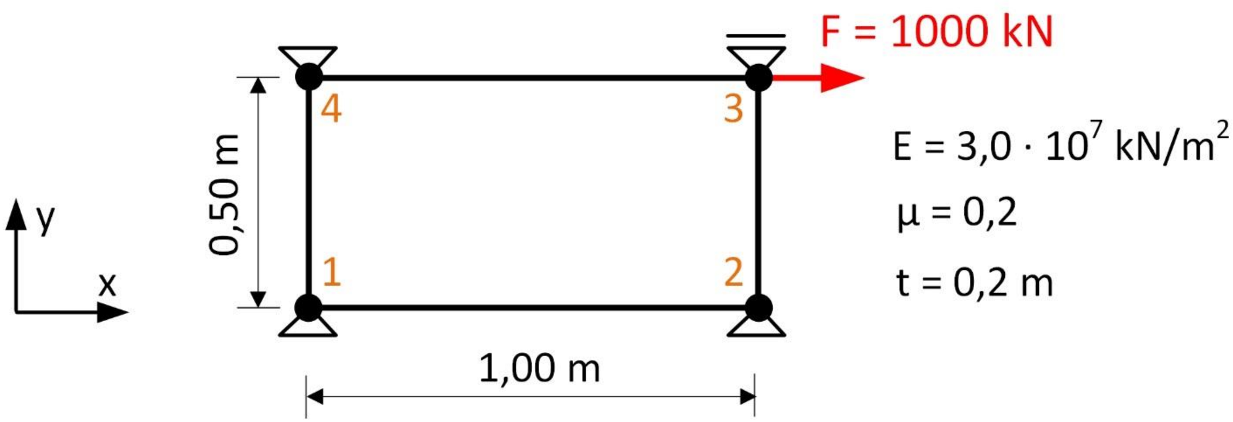
Elements of the stiffness matrix

Element stresses

$$\underline{K}^{(e)}\cdot\underline{u}_e=\underline{F}_e$$

Rectangular plane stress element

Example: Plane stress element with a single free degree of freedom



From the stiffness matrix: $k_{55} \cdot u_3 = F_{x3}$ *with*:

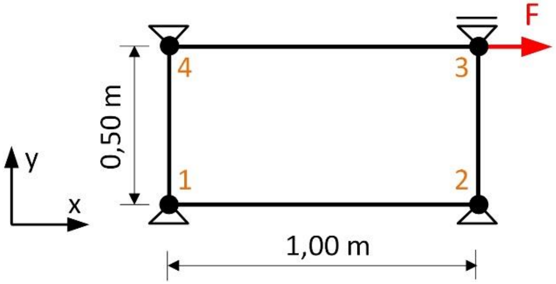
$$k_{55} = \frac{E \cdot t}{12 \cdot (1-\mu^2)} \left(4 \cdot \frac{b}{a} + 2 \cdot (1-\mu) \frac{a}{b} \right) = \frac{3 \cdot 10^7 \cdot 0.2}{12 \cdot (1-0.2^2)} \left(4 \cdot \frac{0.5}{1.0} + 2 \cdot (1-0.2) \frac{1.0}{0.5} \right) = 2.70 \cdot 10^6 \text{ [kN/m]}$$

$$u_3 = F_{x3} / k_{55} = 1000 / 2.7 \cdot 10^6 = 3.69 \cdot 10^{-4} \text{ [m]}$$

Rectangular plane stress element

Example: Plane stress element with a single free degree of freedom

Solutions:
Restraint forces in [kN]



$$F_{x1} = k_{15} \cdot u_3 = 5.21 \cdot 10^5 \left(-2 \cdot \frac{0.5}{1.0} - (1-0.2) \cdot \frac{1.0}{0.5} \right) \cdot 3.69 \cdot 10^{-4} = -500$$

$$F_{y1} = k_{25} \cdot u_3 = 5.21 \cdot 10^5 \left(-\frac{3}{2} \cdot (1+0.2) \cdot \frac{1.0}{0.5} \right) \cdot 3.69 \cdot 10^{-4} = -346$$

$$F_{x2} = k_{35} \cdot u_3 = 5.21 \cdot 10^5 \left(2 \cdot \frac{0.5}{1.0} - 2 \cdot (1-0.2) \cdot \frac{1.0}{0.5} \right) \cdot 3.69 \cdot 10^{-4} = -423$$

$$F_{y2} = k_{45} \cdot u_3 = 5.21 \cdot 10^5 \left(\frac{3}{2} \cdot (1-3 \cdot 0.2) \right) \cdot 3.69 \cdot 10^{-4} = 115$$

$$F_{y3} = k_{65} \cdot u_3 = 5.21 \cdot 10^5 \left(\frac{3}{2} \cdot (1+0.2) \cdot \right) \cdot 3.69 \cdot 10^{-4} = 346$$

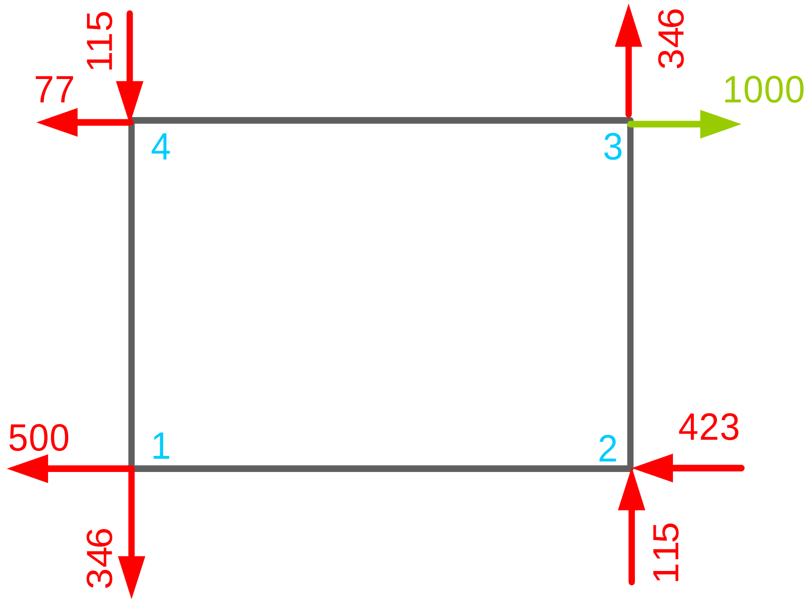
$$F_{x4} = k_{75} \cdot u_3 = 5.21 \cdot 10^5 \left(-4 \cdot \frac{0.5}{1.0} + (1-0.2) \cdot \frac{1.0}{0.5} \right) \cdot 3.69 \cdot 10^{-4} = -77$$

$$F_{y4} = k_{85} \cdot u_3 = 5.21 \cdot 10^5 \left(-\frac{3}{2} \cdot (1-3 \cdot 0.2) \right) \cdot 3.69 \cdot 10^{-4} = 115$$

Rectangular plane stress element

Example: Plane stress element with a single free degree of freedom

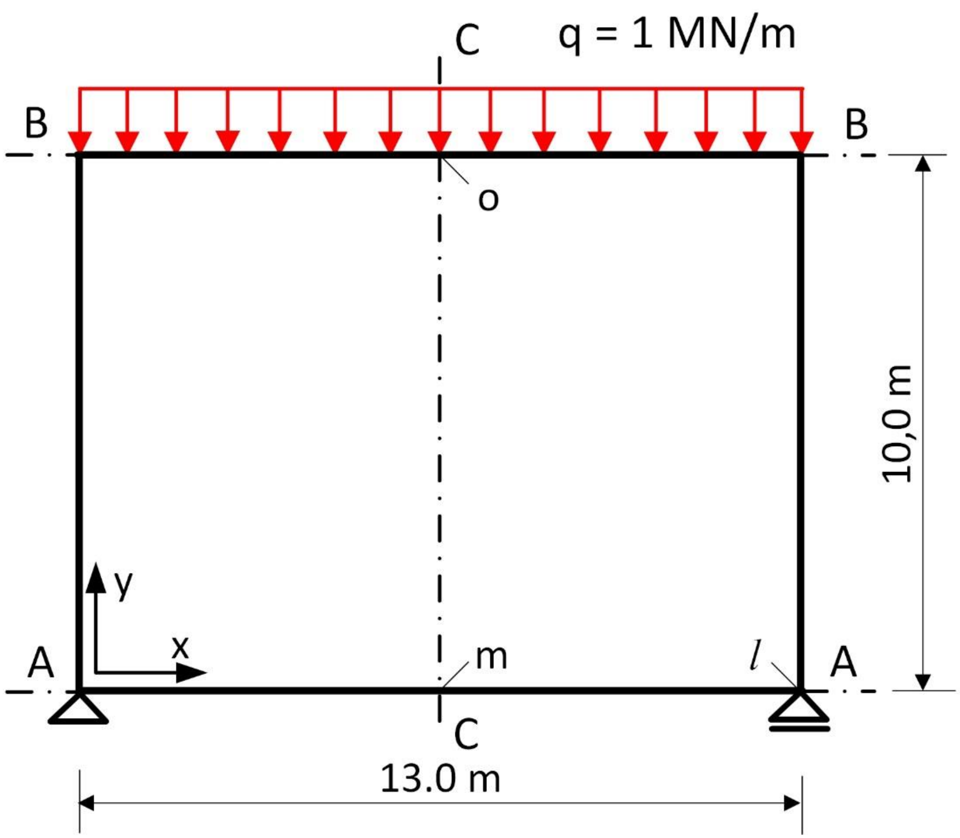
Restraint forces in [kN]



The element forces fulfill the equilibrium conditions!

Rectangular plane stress element

Example: Reinforced concrete deep beam



$E = 3{,}0 \cdot 10^7\ kN/m^2$

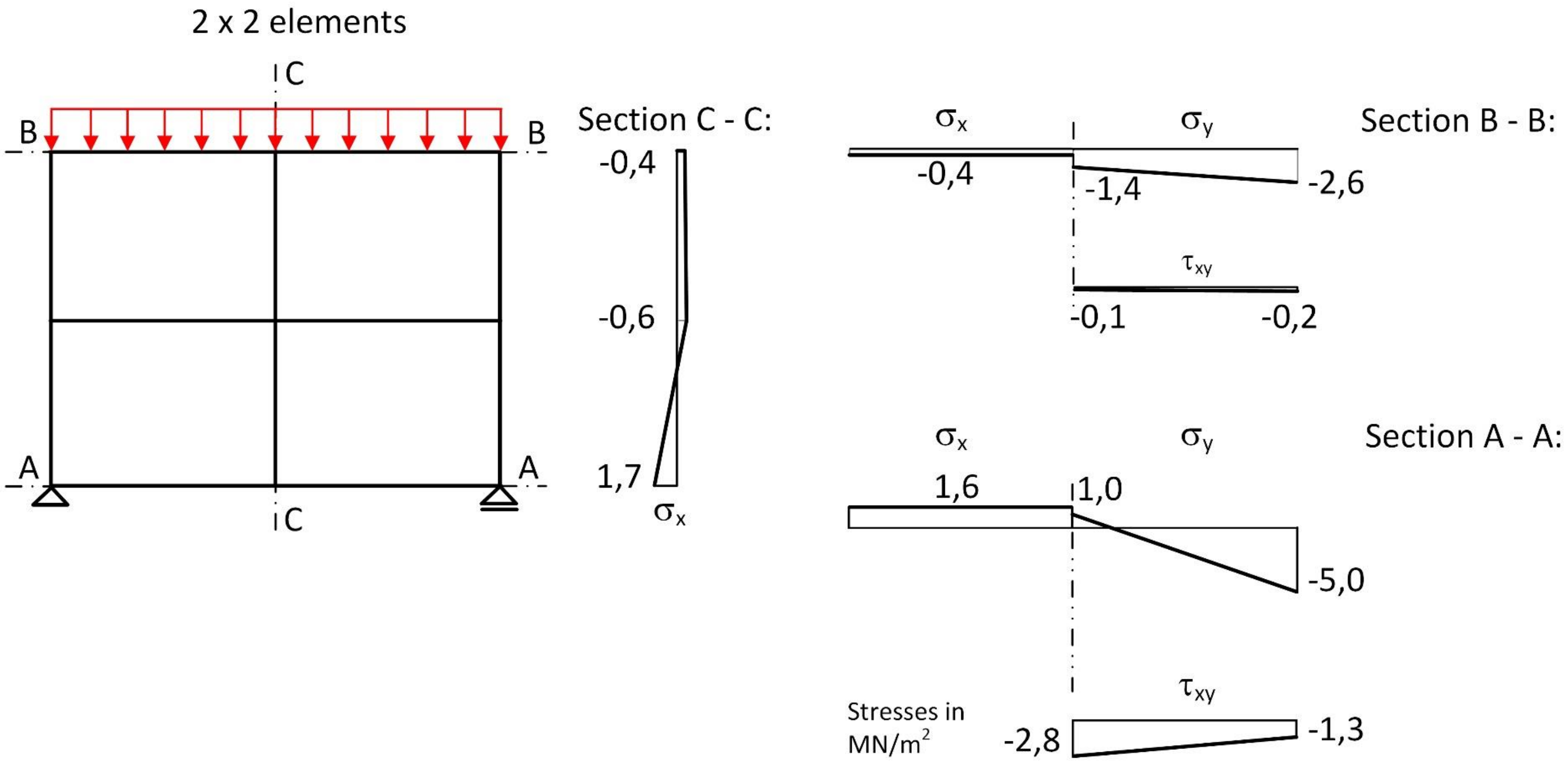
$\mu = 0$

$t = 0{,}5\ m$

Beam theory

Rectangular plane stress element

Example: Reinforced concrete deep beam

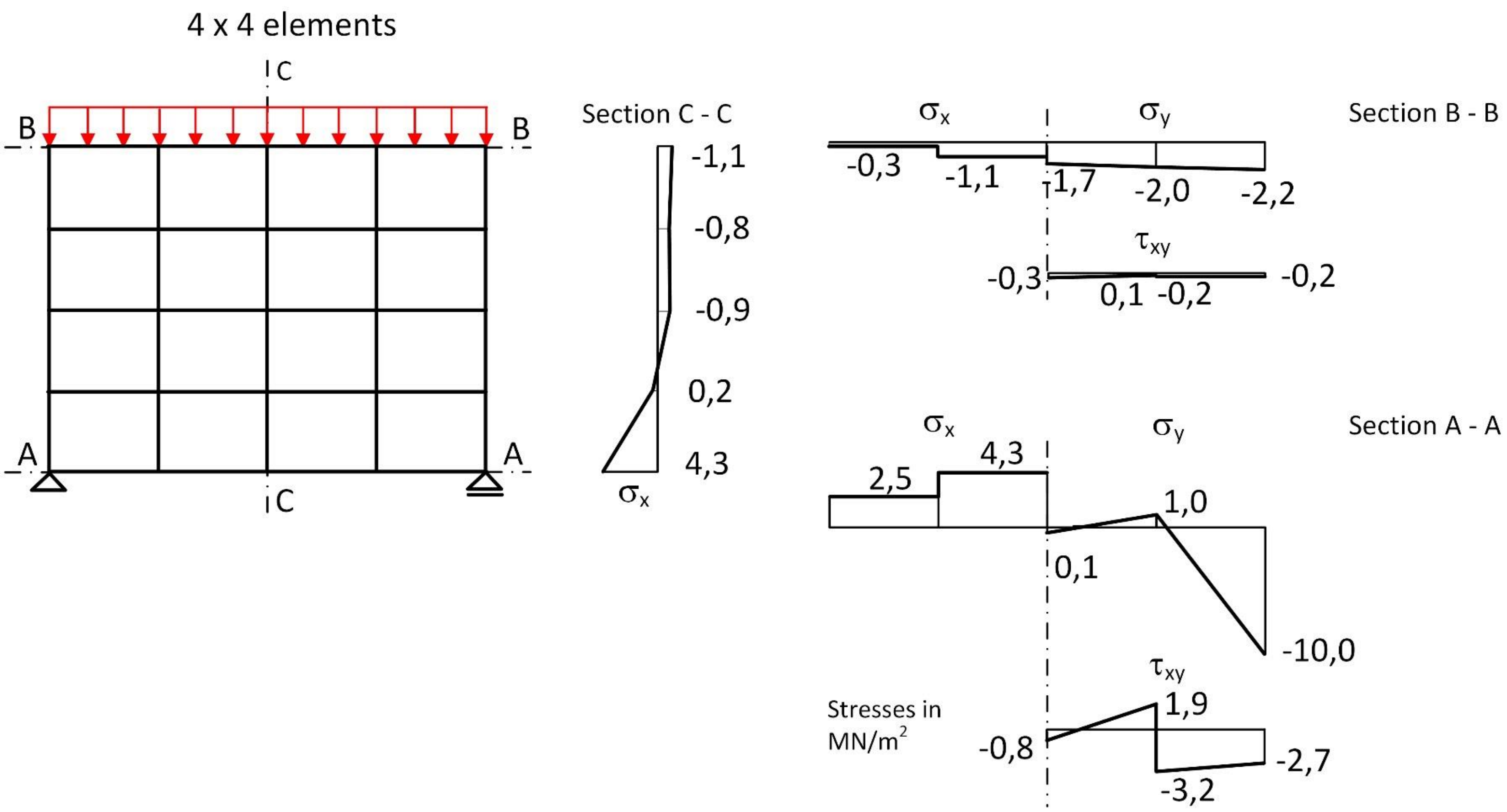


Stresses of the 2x2 FE discretization

Shape functions

Rectangular plane stress element

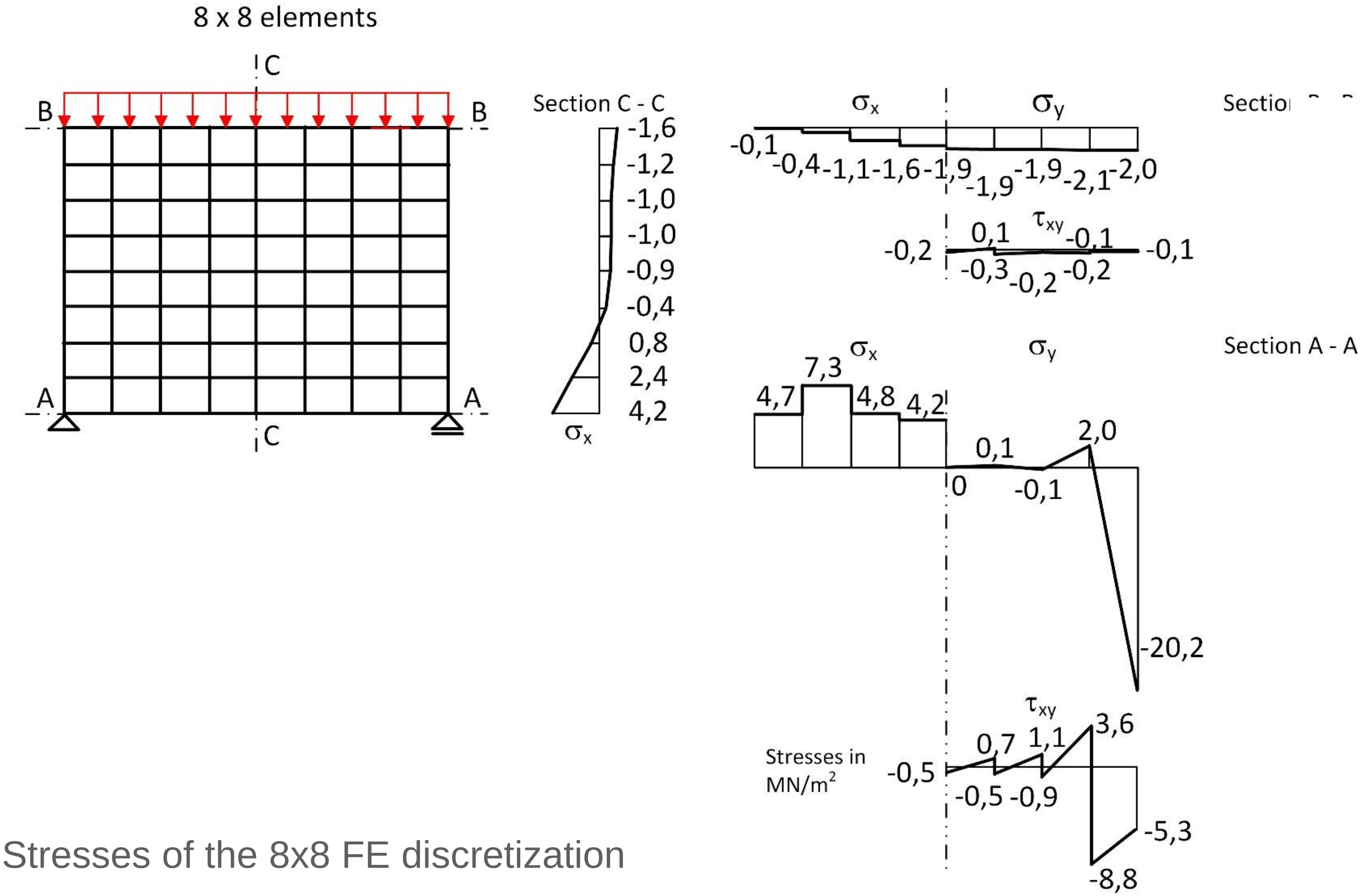
Example: Reinforced concrete deep beam



Stresses of the 4x4 FE discretization

Rectangular plane stress element

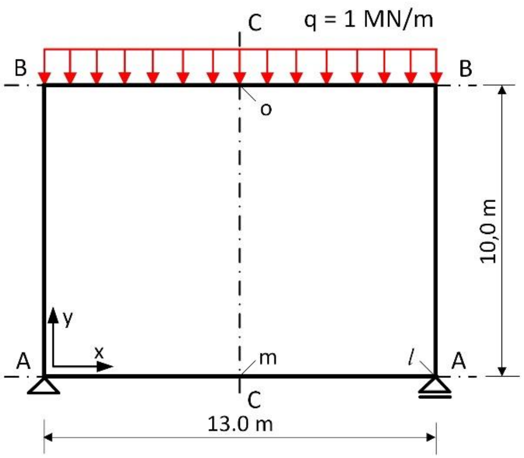
Example: Reinforced concrete deep beam



Stresses of the 8x8 FE discretization

Rectangular plane stress element

Example: Reinforced concrete deep beam



* Values must be discarded due to singularity!

Results:

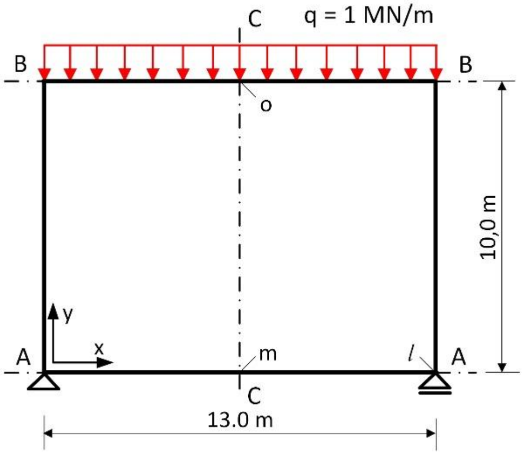
	$\sigma_{x,\ middle,\ top}$	$\sigma_{x,\ middle,\ bottom}$	$\sigma_{y,\ support}$ *	$W_{middle,\ bottom}$ *
2x2	-0,4	1,6	-5,0	$-1{,}22 \cdot 10^{-5}$
4x4	-1,1	4,3	-10,0	$-1{,}76 \cdot 10^{-5}$
8x8	-1,6	4,2	-20,0	$-2{,}40 \cdot 10^{-5}$
16x16	-1,8	4,2	-40,9	$-3{,}05 \cdot 10^{-5}$
32x32	-1,8	4,3	-80,8	$-3{,}69 \cdot 10^{-5}$

$\sigma_{x,\ middle,\ top}$ and $\sigma_{x,\ middle,\ bottom}$ converge to a constant value for mesh refinement.

→ Stresses are reliable for the 8x8 discretization and finer meshes.

Rectangular plane stress element

Example: Reinforced concrete deep beam



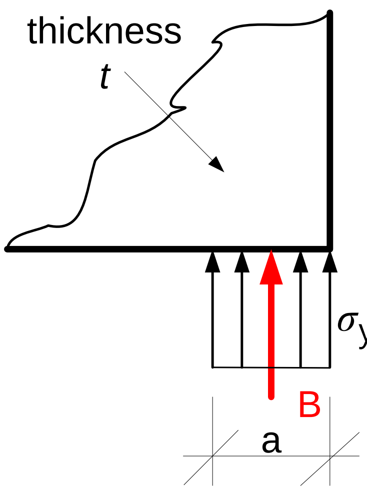
Convergence at the support point:

At the support σ_y increases for a mesh refinement continuously as:

5 → 10 → 20….

This reveals a stress singularity.

The stress values obtained have no physical meaning!



$$\sigma_y = \frac{B}{a \cdot t}$$

Cause

For a point support: a → 0 ➡ $\lim_{a \to 0} \sigma_y = \lim_{a \to 0} \frac{B}{a \cdot t} = \infty$

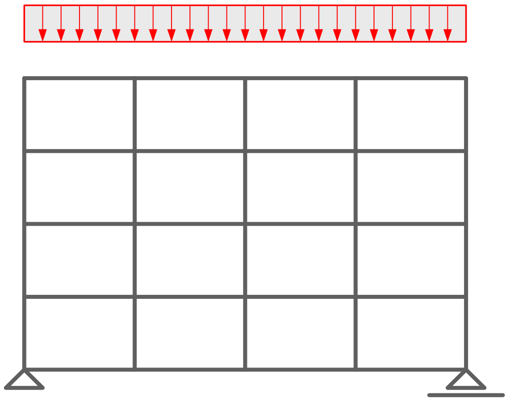
The FE stresses approximate the value ∞ for a mesh refinement.

Other singularities in the model:

- Shear stresses at the support point.
- Vertical displacements

Rectangular plane stress element

Element loads



Equivalent nodal forces for element loads

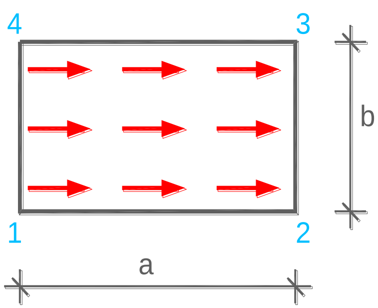
The equivalent nodal forces of an element load are those forces which perform with the virtual nodal displacements the same (virtual external) work as the element loads with their corresponding virtual displacements.

Rectangular plane stress element

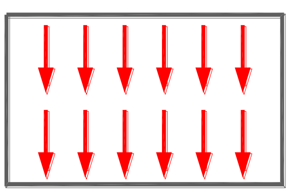
Element loads

Distributed loads

Distributed load p_x



Distributed load py



Forces

$$\overline{W}_a = \int \bar{u} \cdot p_x \; dx\, dy + \int \bar{v} \cdot p_y \; dx\, dy$$

$$\overline{W}_a = \int (\bar{u} \cdot p_x + \bar{v} \cdot p_y) \; dx\, dy = \int \begin{bmatrix} \bar{u} & \bar{v} \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \end{bmatrix} dx\, dy$$

$$= \int \underline{\bar{u}}^T \cdot \underline{p} \; dx\, dy \quad \text{mit} \quad \underline{\bar{u}} = \underline{N} \cdot \underline{\bar{u}}_e \Rightarrow \underline{\bar{u}}^T = \underline{\bar{u}}_e^T \cdot \underline{N}^T$$

$$\overline{W}_a = \underline{\bar{u}}_e^T \cdot \int \underline{N}^T \cdot p \; dx\, dy$$

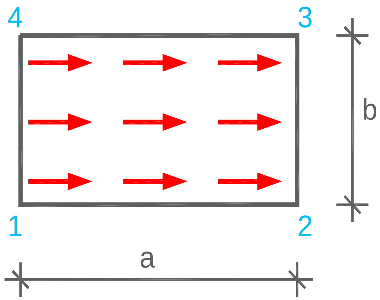
Shape functions of the displacements

Rectangular plane stress element

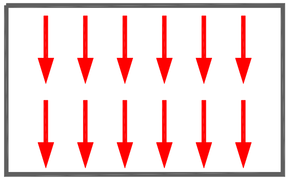
Element loads

Distributed loads

Distributed load p_x



Distributed load py



$$\overline{W}_a = \overline{\underline{u}}_e^T \cdot \underline{F}_L$$

External work

$$\overline{\underline{u}}_e^T \cdot \underline{F}_L = \overline{\underline{u}}_e^T \cdot \int \underline{N}^T \cdot \underline{p} \; dx\, dy$$

Equivalent nodal loads

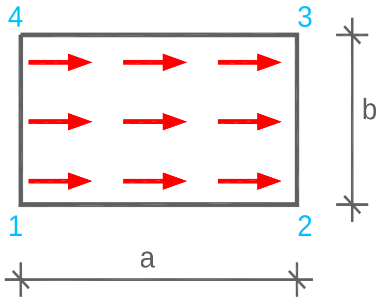
$$\underline{F}_L = \int \underline{N}^T \cdot \underline{p} \; dx\, dy$$

Rectangular plane stress element

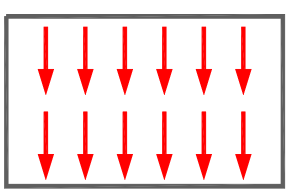
Element loads

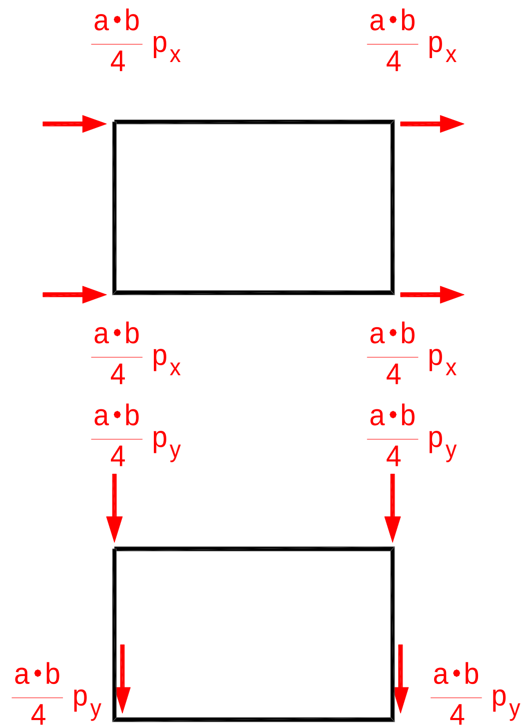
Constant distributed load

Distributed load p_x



Distributed load p_y





$$\underline{F}_L = \int \underline{N}^T \cdot \underline{p}\, dx\, dy$$

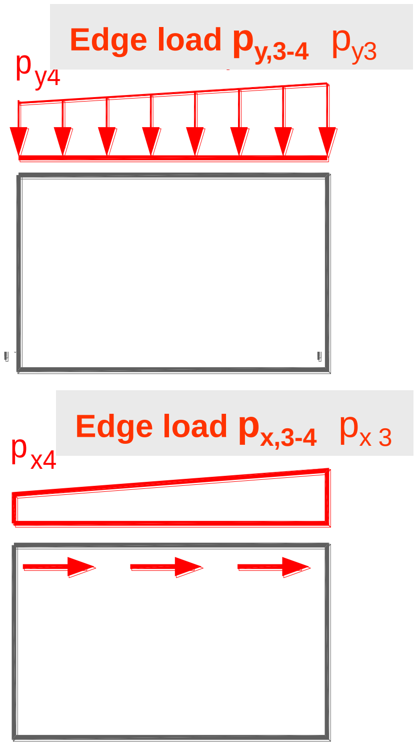
$$\begin{bmatrix} F_{Lx1} \\ F_{Ly1} \\ F_{Lx2} \\ F_{Ly2} \\ F_{Lx3} \\ F_{Ly3} \\ F_{Lx4} \\ F_{Ly4} \end{bmatrix} = \frac{a \cdot b}{4} \begin{bmatrix} p_x \\ p_y \\ p_x \\ p_y \\ p_x \\ p_y \\ p_x \\ p_y \end{bmatrix}$$

Rectangular plane stress element

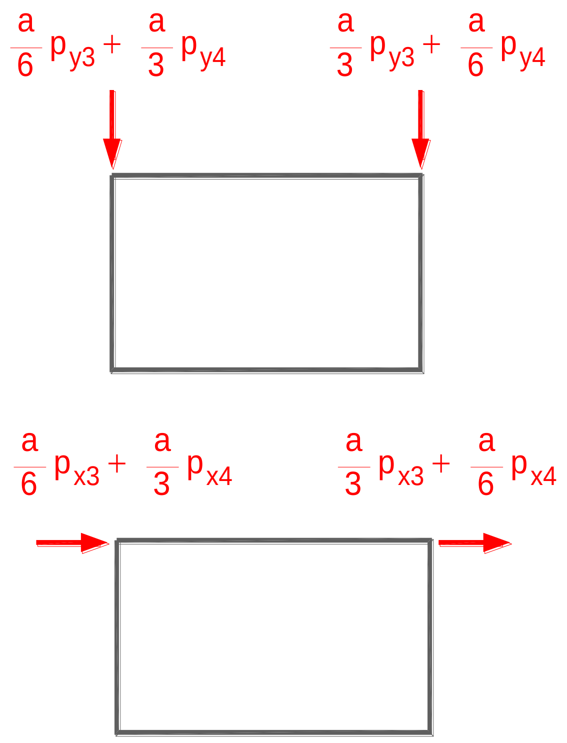
Element loads

Line loads

Element load



Equivalent nodal loads

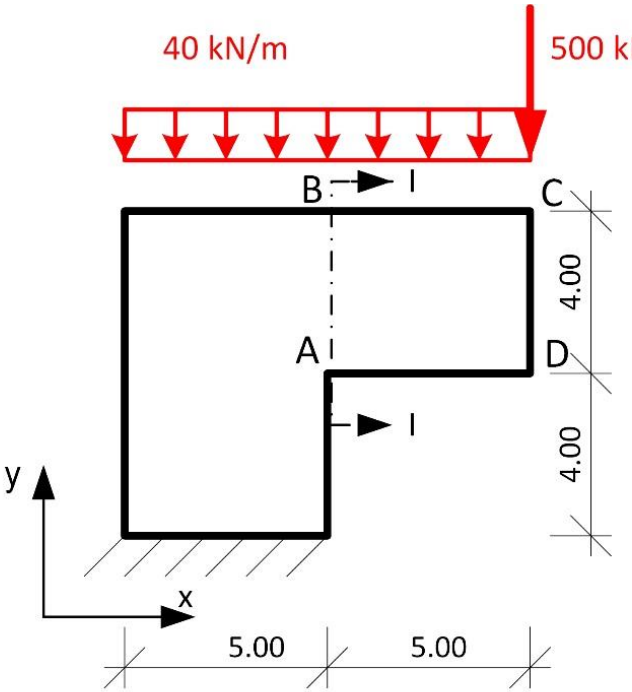


Example

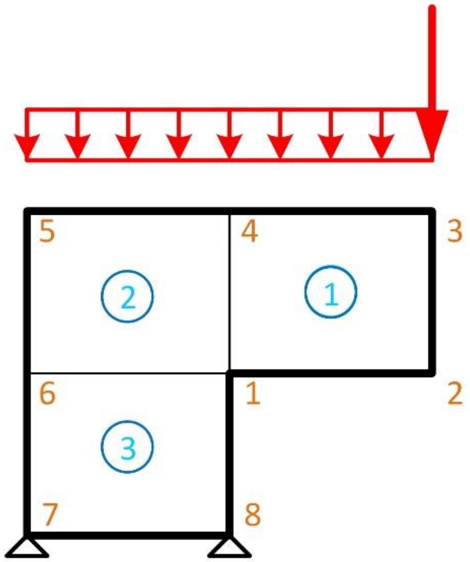
Loading of the upper element edge through linearly distributed loads in x- and y-directions.

Rectangular plane stress element

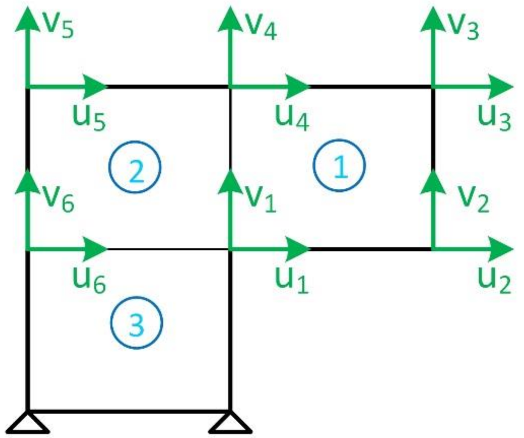
Example: Cantilever structure



Structural model



FE model (very rough)



Stiffness matrix

Rectangular plane stress element

Example: Cantilever structure

Element stiffness matrix

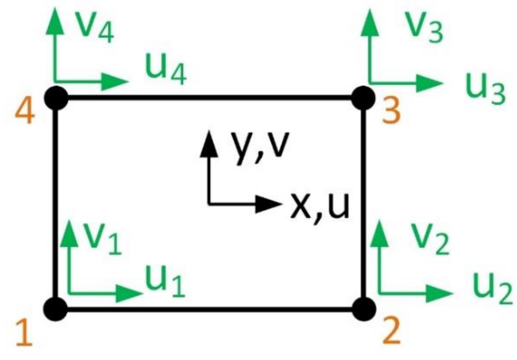
a = 5 m, b = 4 m, t = 0.4 m, E = 3.10^7 kN/m² und μ = 0.2:

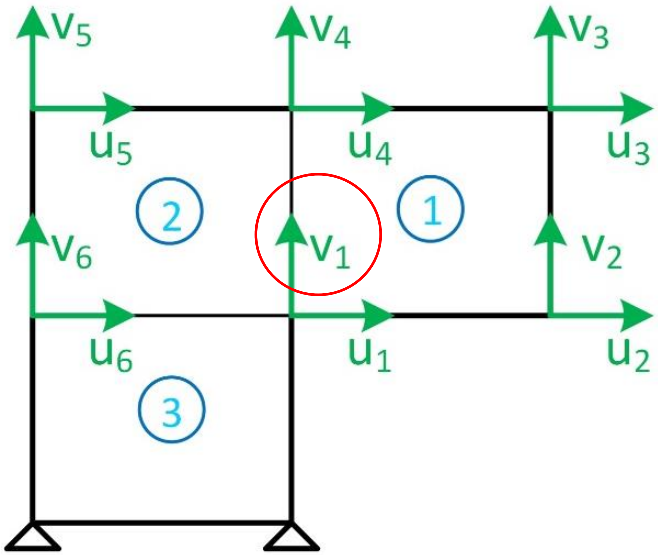
$$\underline{K}^{(e)} = 1.042 \cdot 10^6 \begin{bmatrix} 5.2 & 1.8 & -2.2 & -0.6 & -2.6 & -1.8 & -0.4 & 0.6 \\ 1.8 & 6.28 & 0.6 & 1.22 & -1.8 & -3.14 & -0.6 & -4.36 \\ -2.2 & 0.6 & 5.2 & -1.8 & -0.4 & -0.6 & -2.6 & 1.8 \\ -0.6 & 1.22 & -1.8 & 6.28 & 0.6 & -4.36 & 1.8 & -3.14 \\ -2.6 & -1.8 & -0.4 & 0.6 & 5.2 & 1.8 & -2.2 & -0.6 \\ -1.8 & -3.14 & -0.6 & -4.36 & 1.8 & 6.28 & 0.6 & 1.22 \\ -0.4 & -0.6 & -2.6 & 1.8 & -2.2 & 0.6 & 5.2 & -1.8 \\ 0.6 & -4.36 & 1.8 & -3.14 & -0.6 & 1.22 & -1.8 & 6.28 \end{bmatrix}$$

Rectangular plane stress element

Example: Cantilever structure

Global stiffness matrix





Assembling an entry of the stiffness matrix

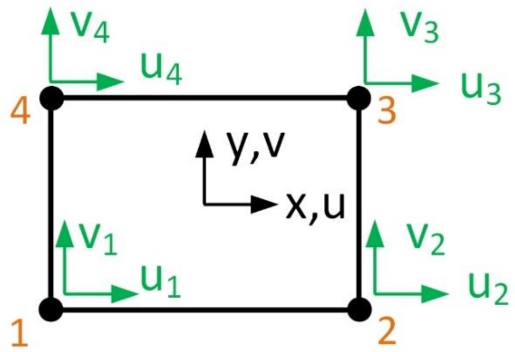
$$k_{2,2}^{(ges)} = k_{2,2}^{(Element 1)} + k_{4,4}^{(Element 2)} + k_{6,6}^{(Element 3)} = 1.042 \cdot 10^6 \cdot (6.28 + 6.28 + 6.28) = 1.042 \cdot 10^6 \cdot 18.84$$

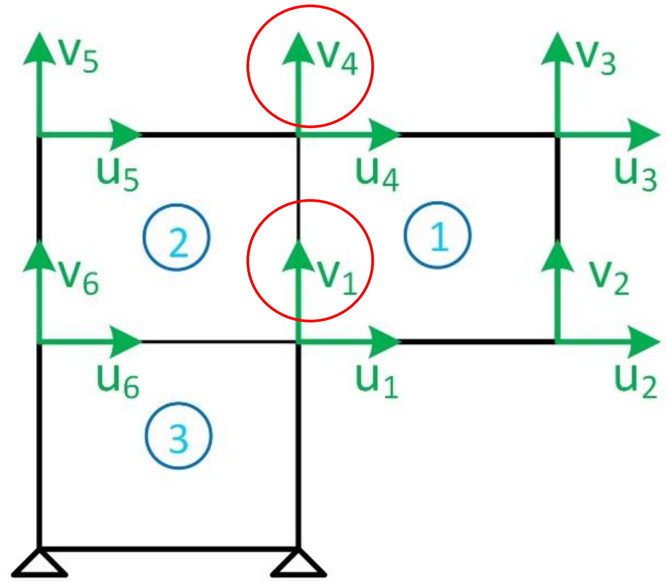
Entries of the stiffness matrix | Stiffness matrix

Rectangular plane stress element

Example: Cantilever structure

Global stiffness matrix





Assembling an entry of the stiffness matrix

$$k_{2,8}^{(ges)} = k_{2,8}^{(Element\,1)} + k_{4,6}^{(Element\,2)} = 1.042 \cdot 10^6 \cdot (-4.36 + (-4.36)) = 1.042 \cdot 10^6 \cdot (-8.72)$$

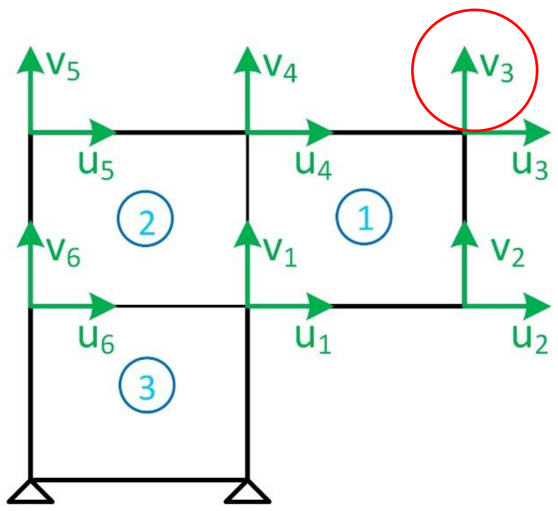
Entries of the stiffness matrix

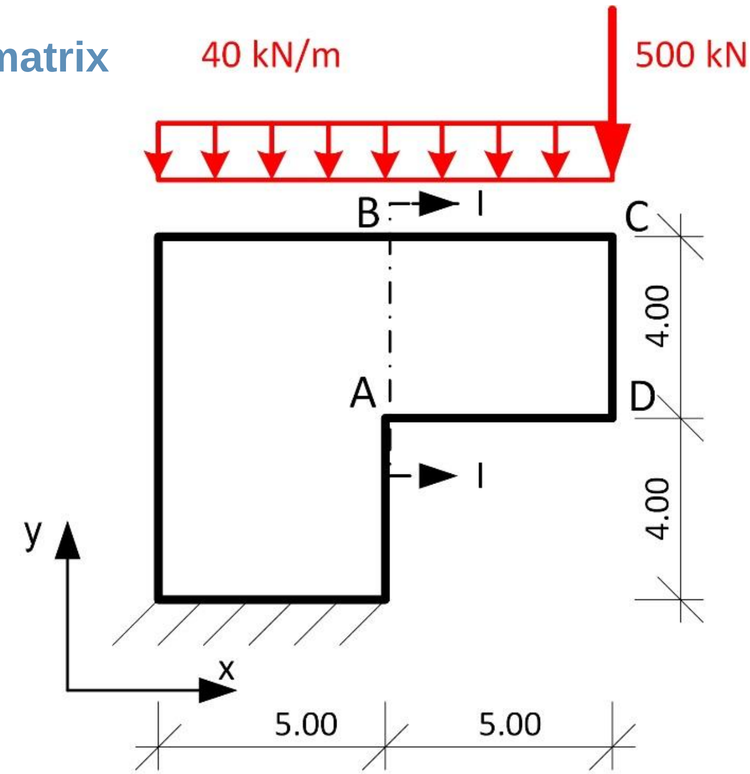
Stiffness matrix

Rectangular plane stress element

Example: Cantilever structure

Global stiffness matrix





Determination of an entry of the load vector,

degree of freedom v_3

$$F_{y,3} = \frac{q \cdot \ell}{2} + F = -\frac{40 \cdot 5}{2} - 500 = -600\,\text{kN}$$

Rectangular plane stress element

Example: Cantilever structure

Global stiffness matrix

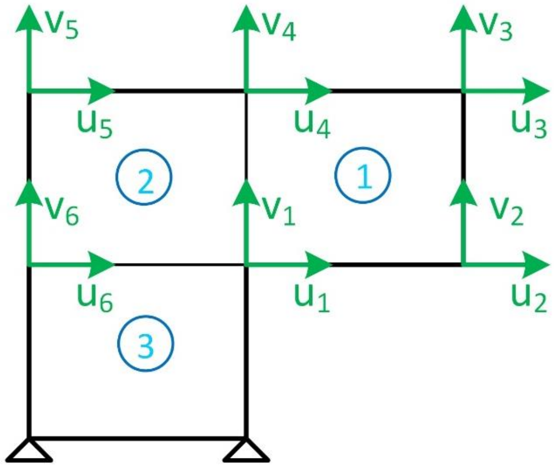
$$
1.042 \cdot 10^{6} \cdot \begin{bmatrix}
15.6 & 1.8 & -2.2 & -0.6 & -2.6 & -1.8 & -0.8 & 0 & -2.6 & 1.8 & -4.4 & 0 \\
1.8 & 18.84 & 0.6 & 1.22 & -1.8 & -3.14 & 0 & -8.72 & 1.8 & -3.14 & 0 & 2.44 \\
-2.2 & 0.6 & 5.2 & -1.8 & -0.4 & -0.6 & -2.6 & 1.8 & 0 & 0 & 0 & 0 \\
-0.6 & 1.22 & -1.8 & 6.28 & 0.6 & -4.36 & 1.8 & -3.14 & 0 & 0 & 0 & 0 \\
-2.6 & -1.8 & -0.4 & 0.6 & 5.2 & 1.8 & -2.2 & -0.6 & 0 & 0 & 0 & 0 \\
-1.8 & -3.14 & -0.6 & -4.36 & 1.8 & 6.28 & 0.6 & 1.22 & 0 & 0 & 0 & 0 \\
-0.8 & 0 & -2.6 & 1.8 & -2.2 & 0.6 & 10.4 & 0 & -2.2 & -0.6 & -2.6 & -1.8 \\
0 & -8.72 & 1.8 & -3.14 & -0.6 & 1.22 & 0 & 12.56 & 0.6 & 1.22 & -1.8 & -3.14 \\
-2.6 & 1.8 & 0 & 0 & 0 & 0 & -2.2 & 0.6 & 5.2 & -1.8 & -0.4 & -0.6 \\
1.8 & -3.14 & 0 & 0 & 0 & 0 & -0.6 & 1.22 & -1.8 & 6.28 & 0.6 & -4.36 \\
-4.4 & 0 & 0 & 0 & 0 & 0 & -2.6 & 1.8 & -0.4 & 0.6 & 10.4 & 0 \\
0 & 2.44 & 0 & 0 & 0 & 0 & -1.8 & -3.14 & -0.6 & -4.36 & 0 & 12.56
\end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -600 \\ 0 \\ -200 \\ 0 \\ -100 \\ 0 \\ 0 \end{bmatrix}
$$

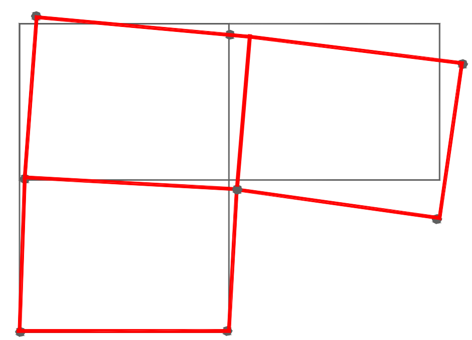
Rectangular plane stress element

Example: Cantilever structure

Displacements

$$\underline{K} \cdot \underline{u} = \underline{F} \quad \Rightarrow \quad u = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0.204 \\ -0.344 \\ 0.080 \\ -1.613 \\ 1.088 \\ -1.635 \\ 0.936 \\ -0.429 \\ 0.818 \\ 0.302 \\ 0.260 \\ 0.237 \end{bmatrix} \cdot 10^{-3} \mathrm{m}$$





Rectangular plane stress element

Example: Cantilever structure

Element stresses

Element stresses will be determined from the nodal displacements.

Example: Element 1, point 3 (local coordinates x=2.5, y=2 with a=5, b=4):

$$\underline{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.2 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.25 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & -0.25 & 0 & 0.25 & 0.2 & 0 & -0.2 \end{pmatrix}$$

$$\underline{D} = 3.125 \cdot 10^7 \cdot \begin{bmatrix} 1 & 0.2 & 0 \\ 0.2 & 1 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$$

$$\underline{u}^{(e)} = \begin{bmatrix} 0.204 \\ -0.344 \\ 0.08 \\ -1.613 \\ 1.088 \\ -1.635 \\ 0.936 \\ -0.429 \end{bmatrix}$$

$$\underline{\sigma} = \underline{D} \cdot \underline{B} \cdot \underline{u}^{(e)} \quad \Rightarrow \quad \underline{\sigma} = \begin{pmatrix} 915.3 \\ 18.1 \\ 137.5 \end{pmatrix} \frac{\text{kN}}{\text{m}^2}$$

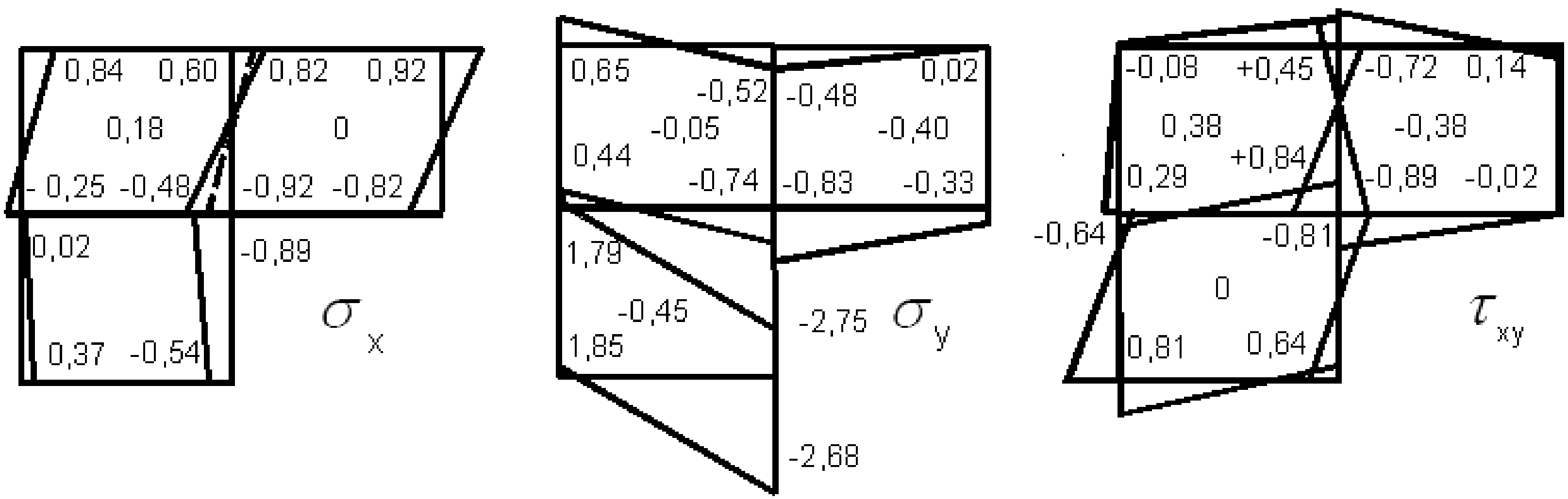
Cantilever structure

Strains

Rectangular plane stress element

Example: Cantilever structure

Element stresses



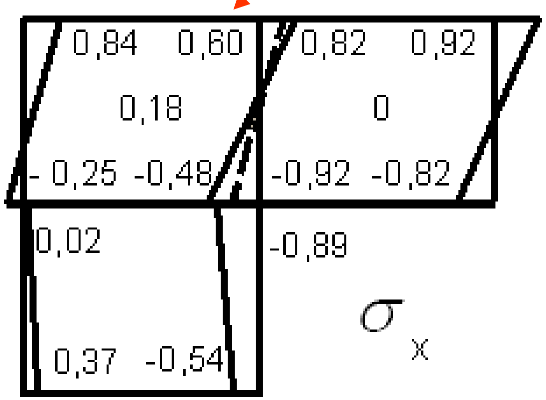
[MN/m²]

Rectangular plane stress element

Example: Cantilever structure

Nodal stresses

Node 4



[MN/m²]

Nodal stresses will be determined as the average value of the element stresses

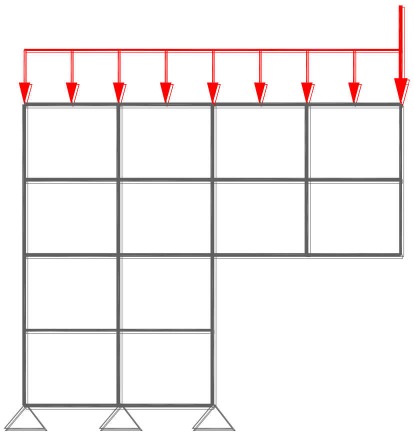
Example: σ_x at node 4

(0,60 + 0,82)/2 = 0.71 MN/m^2

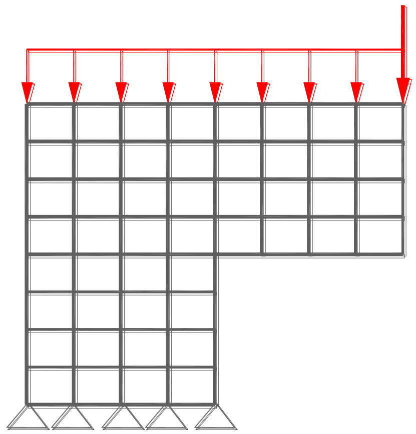
Rectangular plane stress element

Example: Cantilever structure

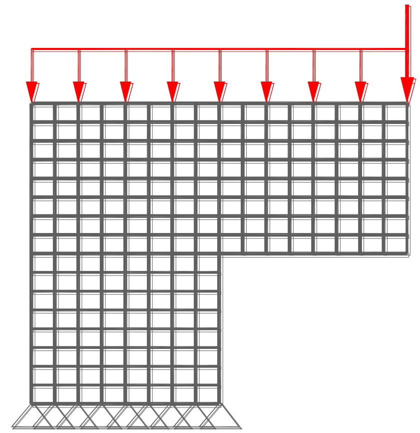
Computation with fine element meshes



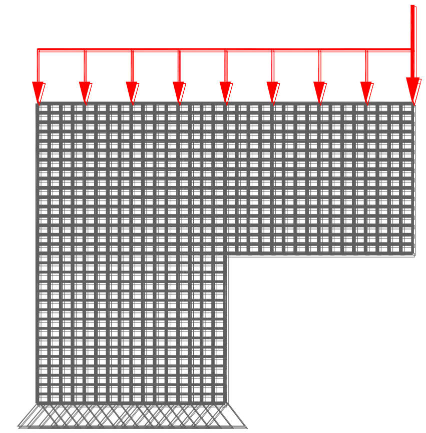
FE model 2(rough)



FE model 3 (rough-middle)



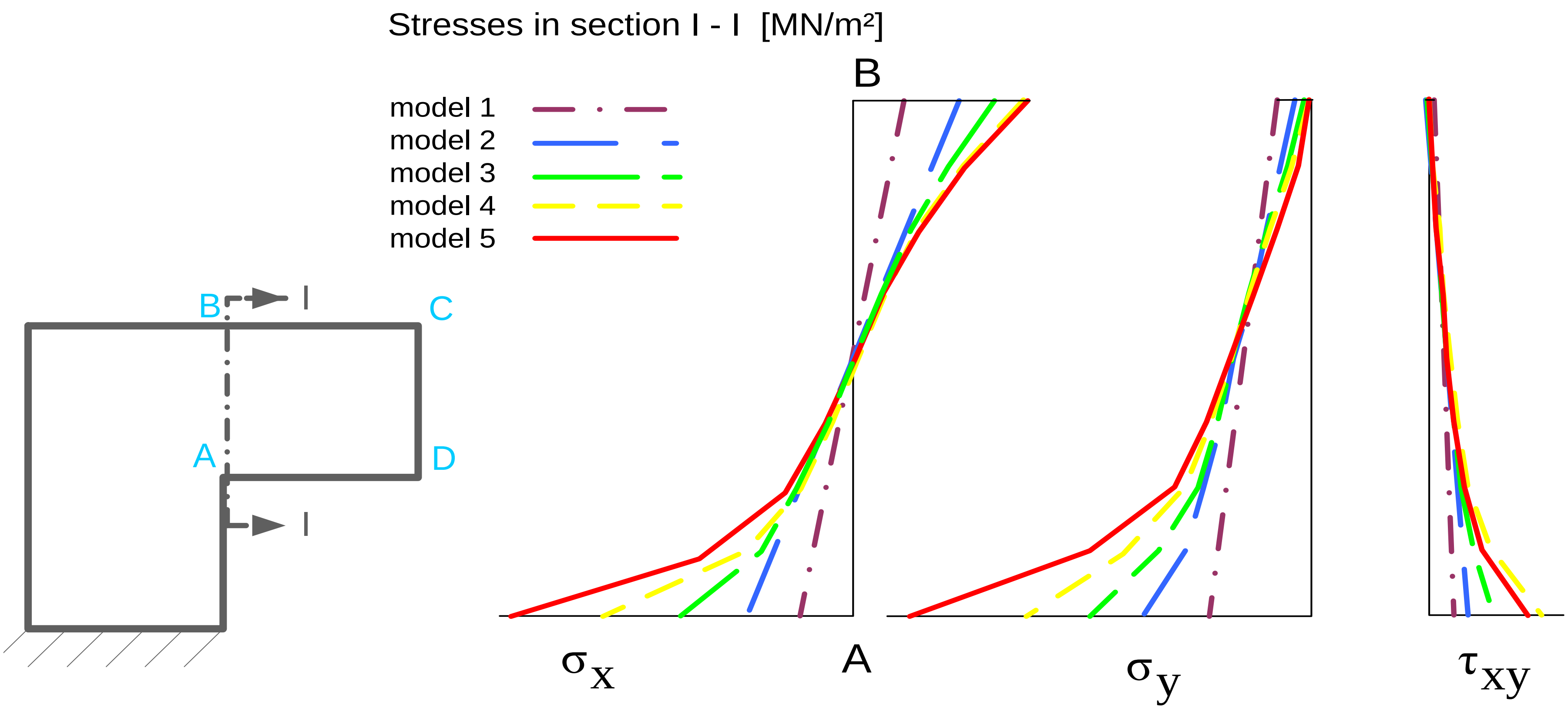
FE model 4 (middle-fine)



FE model 5 (very fine)

Rectangular plane stress element

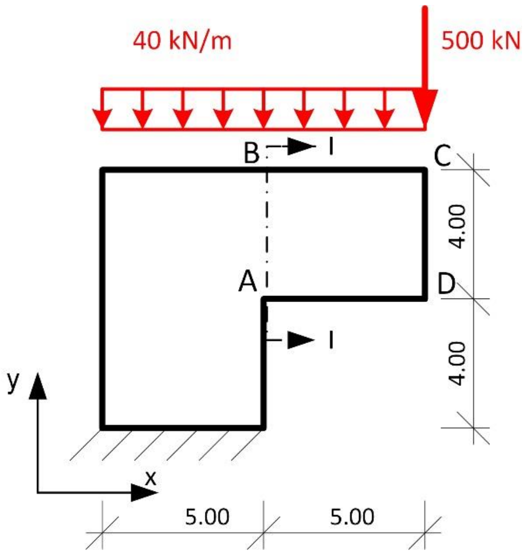
Example: Cantilever structure



Rectangular plane stress element

Example: Cantilever structure

Nodal stresses

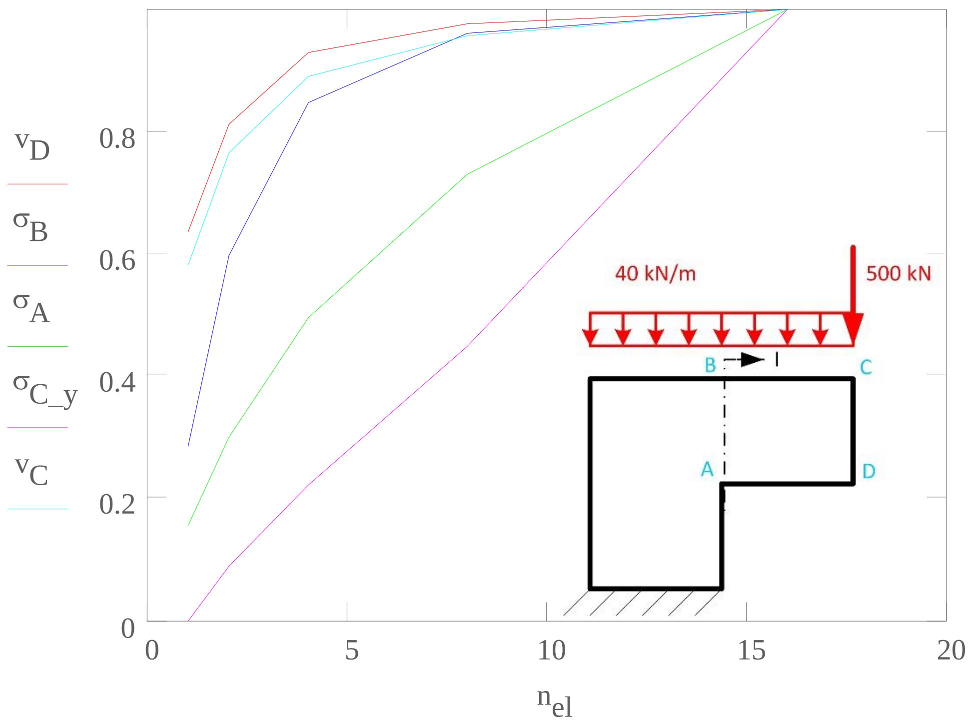


FE- Model*		1	2	3	4	5
FE- size e_x / e_y [m]		5.000/ 4.000	2.500/ 2.000	1.250/ 1.000	0.625/ 0.500	0.3125/ 0.250
Point A	σ_x	-0.761	-1.458	-2.414	-3.560	-5.041
	σ_y	-1.440	-2.178	-3.134	-4.310	-5.847
	τ_{xy}	0.289	0.508	0.996	1.652	2.483
Point B	σ_x	0.707	1.490	2.117	2.397	2.494
	σ_y	-0.502	-0.320	-0.106	-0.063	-0.069
	τ_{xy}	0.135	-0.022	-0.009	0.000	0.002
Point C	σ_x	0.915	0.639	0.412	0.605	1.190
	σ_y	0.018	-0.980	-2.507	-5.068	-10.043
	τ_{xy}	-0.137	0.345	0.666	1.228	2.422
	v_C	-1.64	-2.16	-2.50	-2.69	-2.80
	v_D	-1,61	-2,06	-2,35	-2,48	-2,53

* Stresses in [MN/m²],
Displacements in [mm],
Lengths in [m]

Rectangular plane stress element

Example: Cantilever structure



Convergence behaviour

Reference values set to be „1“
(Solutions of mesh 5):

$$\sigma_{A\text{-}x} = 5.041\,MN/m^2$$

$$\sigma_{B-x} = 2.494\;MN/m^2$$

$$\sigma_{C-y} = 10.043\;MN/m^2$$

$$v_D = 2.53\,mm$$

$$v_C = 2.80\,mm$$

Convergence: $v_{D\text{-}y}$, $v_{C\text{-}y}$, $\sigma_{B\text{-}x}$

Divergence: $\sigma_{C\text{-}y}$, $\sigma_{A\text{-}x}$

Rectangular plane stress element

Example: Cantilever structure

Consequences

- The tension stress σ_x in B converges to the value of 2.5 MN/m^2.
- A sufficient accuracy in section I - I is obtained with more than 6-8 elements over the height.
- At single points e.g. point A (reentrant corner), point C (point load) the stresses increase continuously with a mesh refinement, i.e. they do not converge! This indicates a singularity in the structural model. The displacement at the point load has also a singularity (v_3 in point C).
- The stresses at the nodes obtained by averaging the element stresses have a higher accuracy than the individual element stresses.

End

Introduction

Truss and beam structures

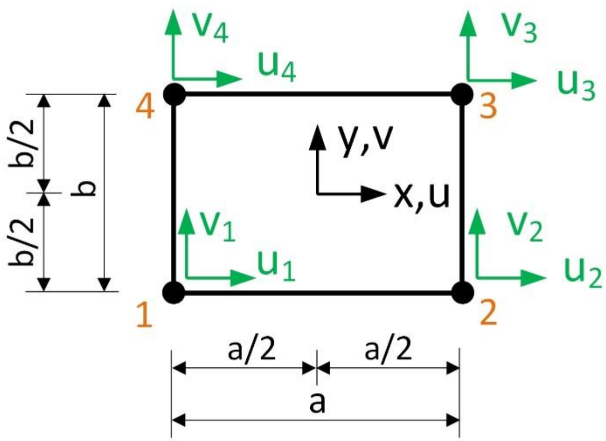
Plate and shell structures

Modeling

Rectangular element for plates in plane stress

Stiffness matrix for the rectangular element for plates in plane stress

with



$$k_{11} = k_{33} = k_{55} = k_{77} = 4\,b/a + 2\,(1-\mu)\,a/b$$
$$k_{22} = k_{44} = k_{66} = k_{88} = 4\,a/b + 2\,(1-\mu)\,b/a$$
$$k_{12} = k_{47} = k_{38} = k_{56} = 3/2\,(1+\mu)$$
$$k_{13} = k_{57} = -4\,b/a + (1-\mu)\,a/b$$
$$k_{14} = k_{27} = k_{58} = k_{36} = -3/2\,(1-3\,\mu)$$
$$k_{15} = k_{37} = -2\,b/a - (1-\mu)\,a/b$$
$$k_{16} = k_{25} = k_{78} = k_{34} = -3/2\,(1+\mu)$$
$$k_{17} = k_{35} = 2\,b/a - 2\,(1-\mu)\,a/b$$
$$k_{18} = k_{23} = k_{67} = k_{45} = 3/2\,(1-3\,\mu)$$
$$k_{24} = k_{68} = 2\,a/b - 2\,(1-\mu)\,b/a$$
$$k_{26} = k_{48} = -2\,a/b - (1-\mu)\,b/a$$
$$k_{28} = k_{46} = -4\,a/b + (1-\mu)\,b/a$$

Rectangular plane stress element

Shape function of the displacements

Shape functions

$$N_1=\frac{1}{4}-\frac{1}{2a}x-\frac{1}{2b}y+\frac{1}{ab}xy$$

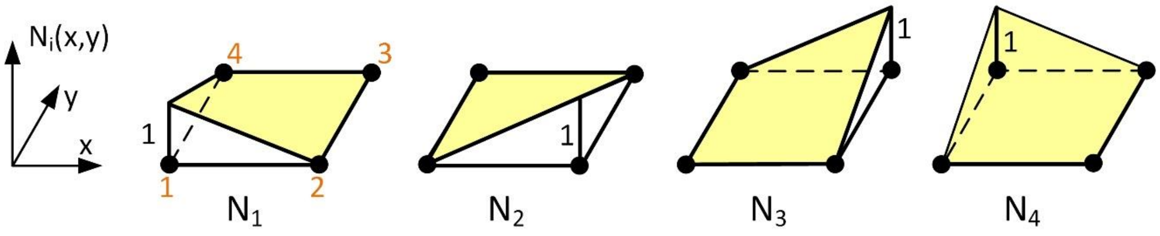
$$N_2=\frac{1}{4}+\frac{1}{2a}x-\frac{1}{2b}y-\frac{1}{ab}xy$$

$$N_3=\frac{1}{4}+\frac{1}{2a}x+\frac{1}{2b}y+\frac{1}{ab}xy$$

$$N_4=\frac{1}{4}-\frac{1}{2a}x+\frac{1}{2b}y-\frac{1}{ab}xy$$

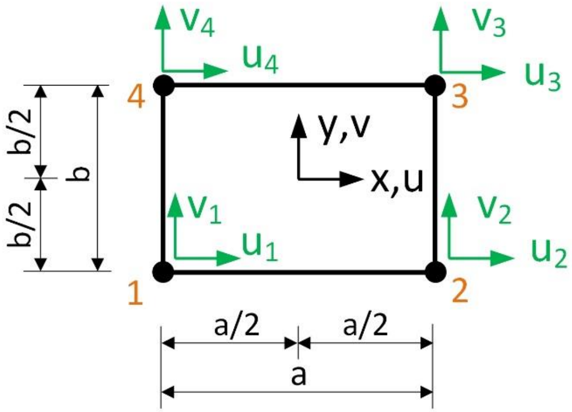
$$\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$\underline{u}=\underline{N}\cdot\underline{u}_e$$



Rectangular plane stress element

Element stresses



$$\underline{\sigma} = \underline{D} \cdot \underline{B} \cdot \underline{u}_e$$

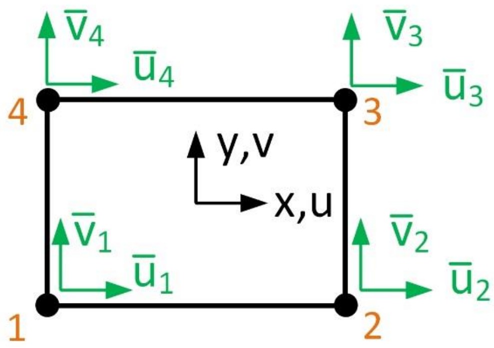
$$\sigma_x = \frac{E}{(1-\mu^2)\cdot 2\cdot a\cdot b}\cdot[(2\cdot y-b)\cdot(u_1-u_2)+(2\cdot y+b)\cdot(u_3-u_4)+ \mu\cdot((2\cdot x-a)\cdot(v_1-v_4)+(2\cdot x+a)\cdot(-v_2+v_3))]$$

$$\sigma_y = \frac{E}{(1-\mu^2)\cdot 2\cdot a\cdot b}\cdot[\mu\cdot((2\cdot y-b)\cdot(u_1-u_2)+(2\cdot y+b)\cdot(u_3-u_4))+ (2\cdot x-a)\cdot(v_1-v_4)+(2\cdot x+a)\cdot(-v_2+v_3)]$$

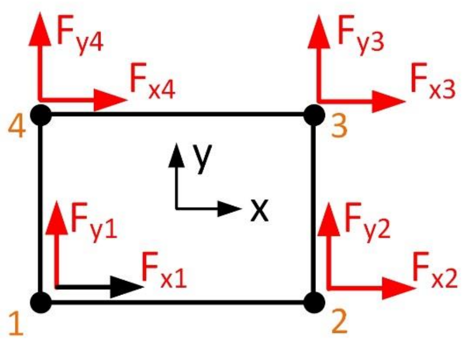
$$\tau_{xy} = \frac{E}{4\cdot(1+\mu)\cdot a\cdot b}\cdot[(2\cdot y-b)\cdot(v_1-v_2)+(2\cdot y+b)\cdot(v_3-v_4)+ (2\cdot x-a)\cdot(u_1-u_4)+(2\cdot x+a)\cdot(u_3-u_2)]$$

Rectangular plane stress element

Principle of virtual displacements



Virtual displacements



Real forces

External work

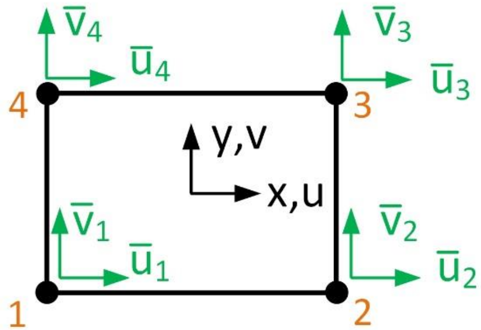
done by the element nodal forces:

$$\overline{W}_a = \underline{\bar{u}}_e^T \cdot \underline{F}_e$$

$$\overline{W}_a = \begin{bmatrix} \bar{u}_1 & \bar{v}_1 & \bar{u}_2 & \bar{v}_2 & \bar{u}_3 & \bar{v}_3 & \bar{u}_4 & \bar{v}_4 \end{bmatrix} \cdot \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}$$

Rectangular plane stress element

Principle of virtual displacements



Virtual displacements

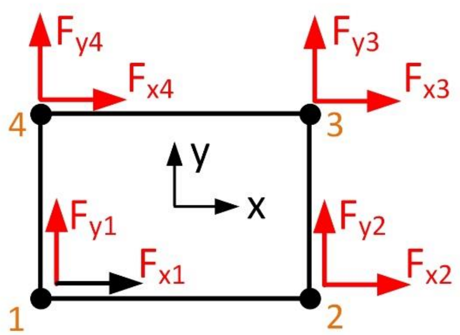
$$\overline{W}_a = \underline{\bar{u}}_e^T \cdot \underline{F}_e \qquad \overline{W}_i = \underline{\bar{u}}_e^T \cdot t \int \underline{B}^T \cdot \underline{D} \cdot \underline{B} \; dx \; dy \cdot \underline{u}_e$$

$$\overline{W}_i = \overline{W}_a$$

$$\underline{\bar{u}}_e^T \cdot t \int \underline{B}^T \cdot \underline{D} \cdot \underline{B} \; dx \; dy \cdot \underline{u}_e = \underline{\bar{u}}_e^T \cdot \underline{F}_e$$

This applies to all virtual displacements $\underline{\bar{u}}_e$

$$\rightarrow \quad t \cdot \int \underline{B}^T \cdot \underline{D} \cdot \underline{B} \; dx \; dy \cdot \underline{u}_e = \underline{F}_e$$

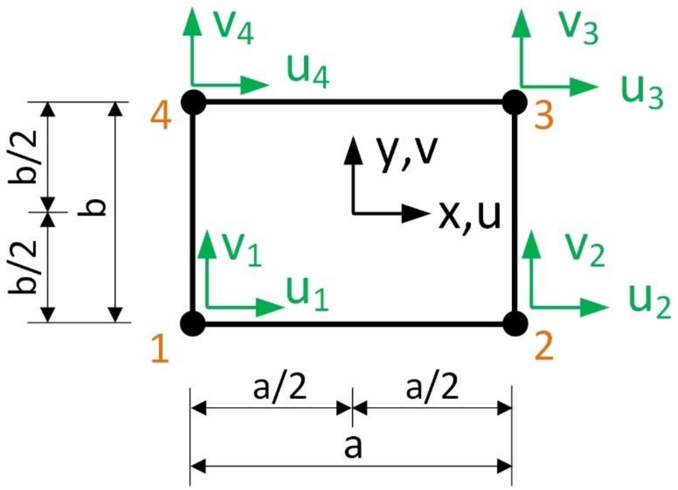


Real forces

$$\underline{K}^{(e)} \cdot \underline{u}_e = \underline{F}_e \qquad \underline{K}^{(e)} = t \cdot \int \underline{B}^T \cdot \underline{D} \cdot \underline{B} \; dx \; dy$$

Rectangular plane stress element

Element displacements



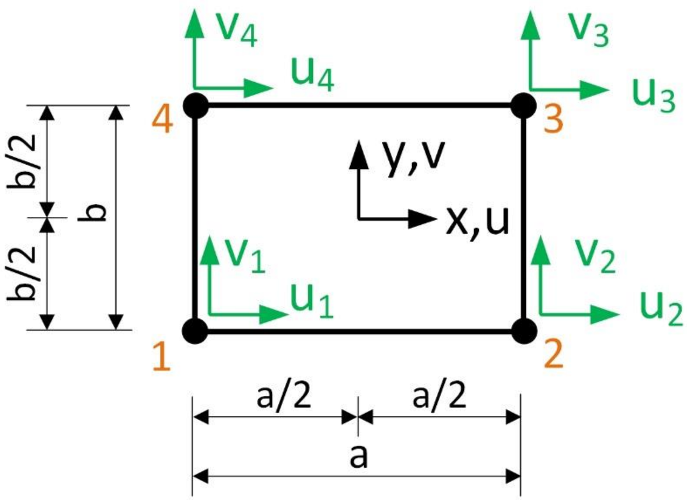
$$\underline{u} = \underline{N} \cdot \underline{u}_e$$

$$u(x,y) = \frac{1}{4}(u_1 + u_2 + u_3 + u_4) + x \cdot \frac{1}{2a}(-u_1 + u_2 + u_3 - u_4) + y\frac{1}{2b}(-u_1 - u_2 + u_3 + u_4) + x \cdot y \cdot \frac{1}{ab}(u_1 - u_2 + u_3 - u_4)$$

$$v(x,y) = \frac{1}{4}(v_1 + v_2 + v_3 + v_4) + x \cdot \frac{1}{2a}(-v_1 + v_2 + v_3 - v_4) + y\frac{1}{2b}(-v_1 - v_2 + v_3 + v_4) + x \cdot y \cdot \frac{1}{ab}(v_1 - v_2 + v_3 - v_4)$$

Rectangular plane stress element

Element displacements



$$\underline{u} = \underline{N} \cdot \underline{u}_e$$

$$u(x,y) = \frac{1}{4}(u_1 + u_2 + u_3 + u_4) + x \cdot \frac{1}{2a}(-u_1 + u_2 + u_3 - u_4) + y\frac{1}{2b}(-u_1 - u_2 + u_3 + u_4) + x \cdot y \cdot \frac{1}{ab}(u_1 - u_2 + u_3 - u_4)$$

$$v(x,y) = \frac{1}{4}(v_1 + v_2 + v_3 + v_4) + x \cdot \frac{1}{2a}(-v_1 + v_2 + v_3 - v_4) + y\frac{1}{2b}(-v_1 - v_2 + v_3 + v_4) + x \cdot y \cdot \frac{1}{ab}(v_1 - v_2 + v_3 - v_4)$$

Rectangular plane stress element

Strains

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}=\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\end{bmatrix}=\begin{bmatrix}\frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0\\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x}\end{bmatrix}\cdot\begin{bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\end{bmatrix}$$

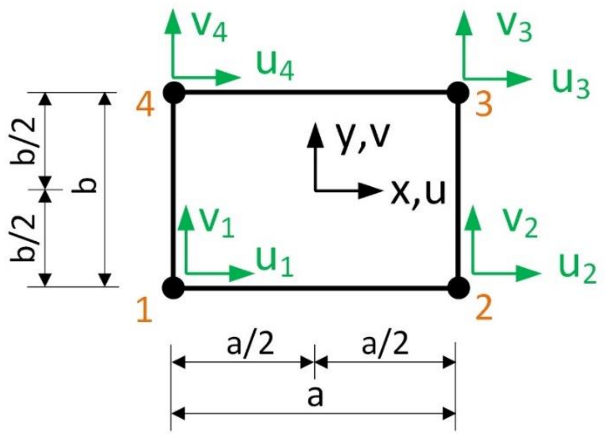
$$\underline{\varepsilon} = \underline{B} \cdot \underline{u}_e$$

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}=\frac{1}{2ab}\begin{bmatrix}2y-b & 0 & -2y+b & 0 & 2y+b & 0 & -2y-b & 0\\ 0 & 2x-a & 0 & -2x-a & 0 & 2x+a & 0 & -2x+a\\ 2x-a & 2y-b & -2x-a & -2y+b & 2x+a & 2y+b & -2x+a & -2y-b\end{bmatrix}\cdot\begin{bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\end{bmatrix}$$

Rectangular plane stress element

Stiffness matrix of a rectangular plate element

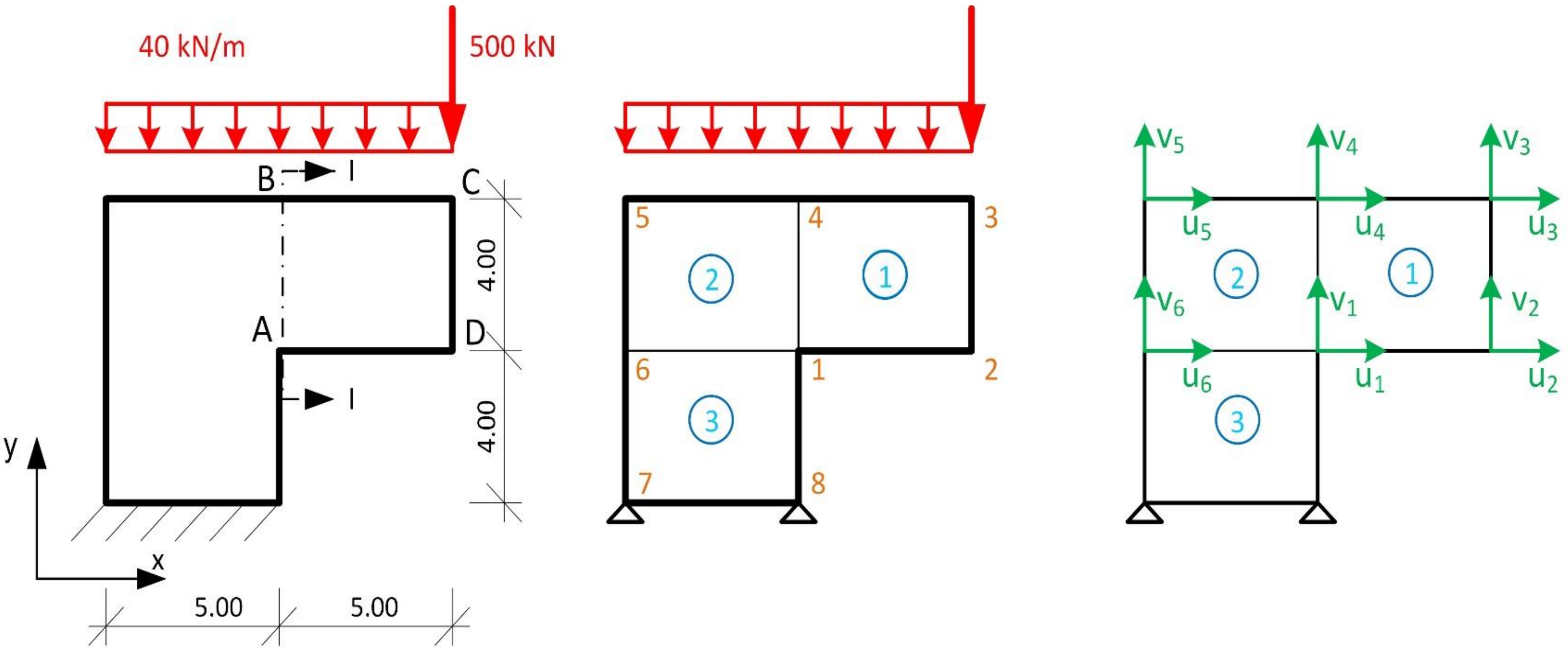
$$\frac{E \cdot t}{12 \cdot (1-\mu^2)} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} & k_{57} & k_{58} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} & k_{67} & k_{68} \\ k_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} & k_{78} \\ k_{81} & k_{82} & k_{83} & k_{84} & k_{85} & k_{86} & k_{87} & k_{88} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \\ F_{x4} \\ F_{y4} \end{bmatrix}$$



$$\underline{K}^{(e)} \cdot \underline{u}_e = \underline{F}_e$$

Rectangular plane stress element

Example: Cantilever structure

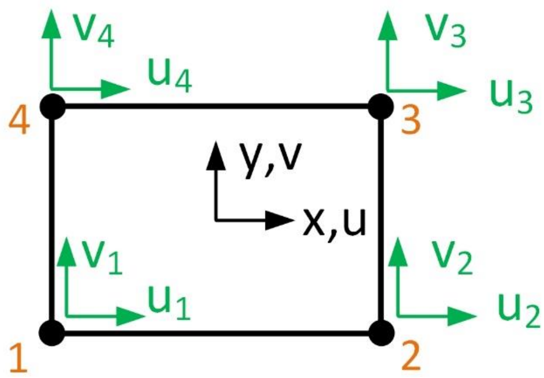


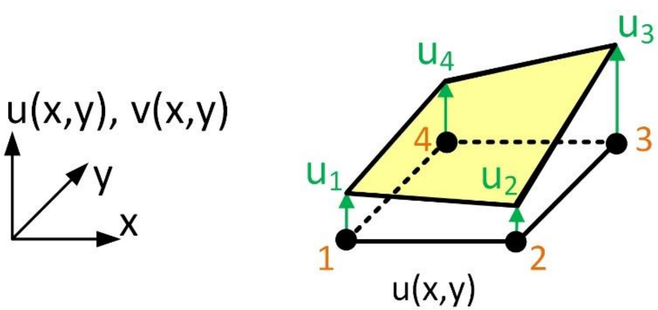
Structural model

FE model (very rough)

Rectangular plane stress element

Shape function of the displacements





Shape functions of **u**

Bilinear shape function for the displacements:

$$u = \alpha_1 + \alpha_2 \cdot x + \alpha_3 \cdot y + \alpha_4 \cdot x \cdot y$$

$$v = \beta_1 + \beta_2 \cdot x + \beta_3 \cdot y + \beta_4 \cdot x \cdot y$$

bilinear term

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

$$\underline{u} = \underline{N}_a \cdot \underline{a}$$

Rectangular plane stress element

Shape function of the displacements

Shape functions

$$N_1=\frac{1}{4}-\frac{1}{2a}x-\frac{1}{2b}y+\frac{1}{ab}xy$$

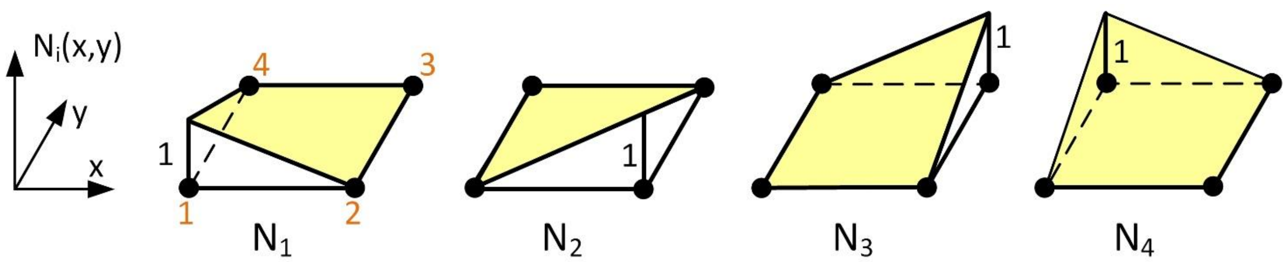
$$N_2=\frac{1}{4}+\frac{1}{2a}x-\frac{1}{2b}y-\frac{1}{ab}xy$$

$$N_3=\frac{1}{4}+\frac{1}{2a}x+\frac{1}{2b}y+\frac{1}{ab}xy$$

$$N_4=\frac{1}{4}-\frac{1}{2a}x+\frac{1}{2b}y-\frac{1}{ab}xy$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$\underline{u}=\underline{N}\cdot\underline{u}_e$$



Rectangular plane stress element

Stresses

Strain vector $\underline{\varepsilon} = \underline{B} \cdot \underline{u}_e$

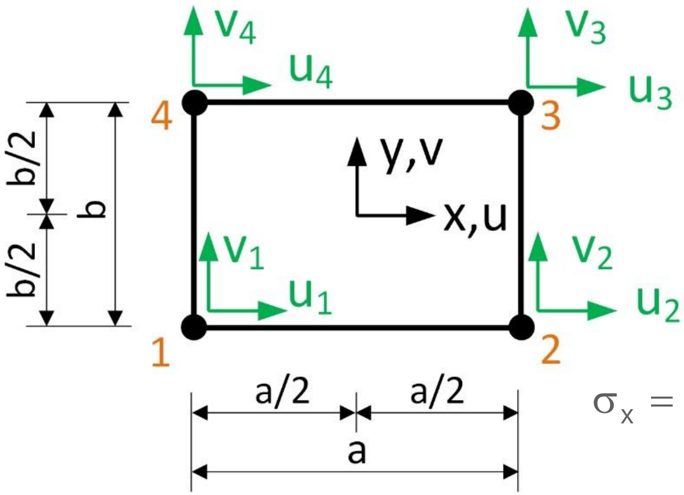
Hooke's law $\underline{\sigma} = \underline{D} \cdot \underline{\varepsilon}$

Stress vector $\underline{\sigma} = \underline{D} \cdot \underline{B} \cdot \underline{u}_e$

with $$\underline{D} = \frac{E}{1-\mu^2} \cdot \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

Rectangular plane stress element

Element stresses



$$\underline{\sigma} = \underline{D} \cdot \underline{B} \cdot \underline{u}_e$$

$$\sigma_x = \frac{E}{(1-\mu^2)\cdot 2\cdot a\cdot b}\cdot[(2\cdot y-b)\cdot(u_1-u_2)+(2\cdot y+b)\cdot(u_3-u_4)+ \mu\cdot((2\cdot x-a)\cdot(v_1-v_4)+(2\cdot x+a)\cdot(-v_2+v_3))]$$

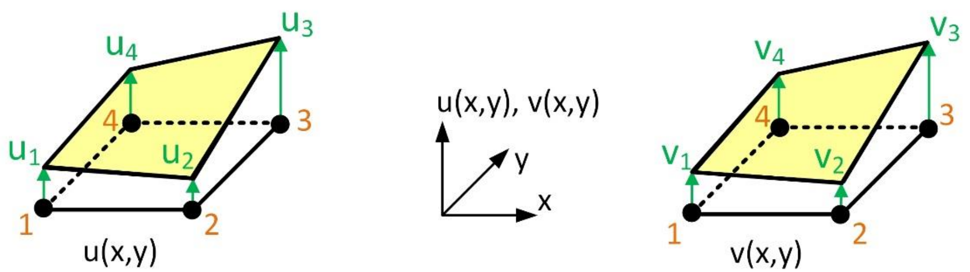
$$\sigma_y = \frac{E}{(1-\mu^2)\cdot 2\cdot a\cdot b}\cdot[\mu\cdot((2\cdot y-b)\cdot(u_1-u_2)+(2\cdot y+b)\cdot(u_3-u_4))+ (2\cdot x-a)\cdot(v_1-v_4)+(2\cdot x+a)\cdot(-v_2+v_3)]$$

$$\tau_{xy} = \frac{E}{4\cdot(1+\mu)\cdot a\cdot b}\cdot[(2\cdot y-b)\cdot(v_1-v_2)+(2\cdot y+b)\cdot(v_3-v_4)+ (2\cdot x-a)\cdot(u_1-u_4)+(2\cdot x+a)\cdot(u_3-u_2)]$$

Rectangular plane stress element

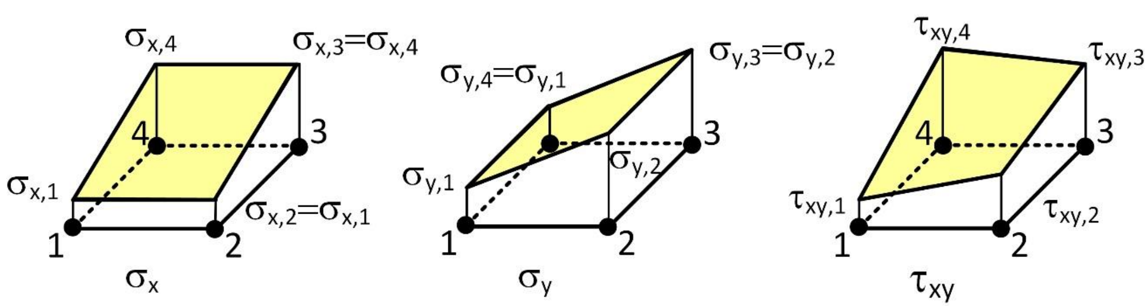
Shape functions of the rectangular plane stress element and Stresses derived thereof

Shape functions



bilinear functions

Stresses derived from the shape functions



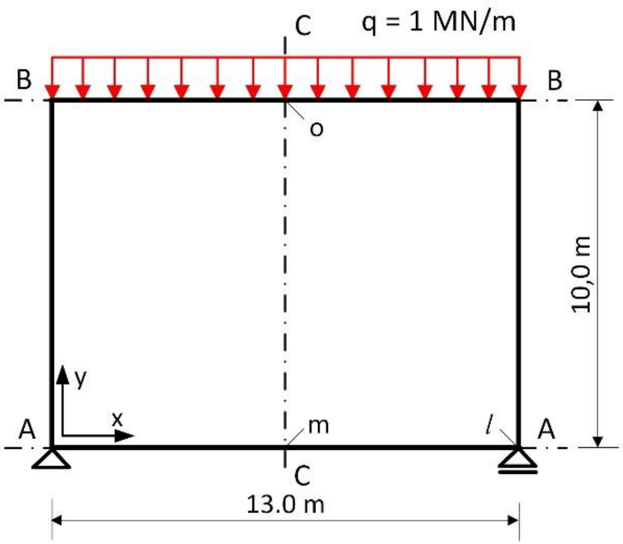
for $\mu = 0$:

σ_x constant in x-direction
linear in y-direction

σ_y constant in y-direction
linear in x-direction

Rectangular plane stress element

Example: Reinforced concrete deep beam



$E = 3{,}0 \cdot 10^7$ [kN/m²]

$\mu = 0{,}0$

$t = 0{,}5$ [m]

Comparison with the beam theory

$M_{max} = ql^2/8 = 13.00^2/8 \qquad = 21.13$ MNm

$W \quad = t\,h^2/6 = 0.5 \cdot 10^2 / 6 \qquad = \; 8.33\ m^3$

$\sigma_{o,u} \quad = + - M / W = + - 21.125 / 8.33 = \; 2.50\ MN/m^2$

The stress value + /- 2.5 MN/m^2 in the beam theory assumes a linear distribution of the stresses.