Finite Elements in Structural Analysis

Introduction

Truss and beam structures

Plate and shell structures

Modeling

Finite elements for plane stress elements

Properties of finite elements

Compulsory requirements of finite elements

a) Rigid body displacements must not provoke nodal forces

b) Elements must be able to represent constant strains and constant stresses exactly

Optional requirements of finite elements

c) Continuity of the displacements

d) Geometric isotropy

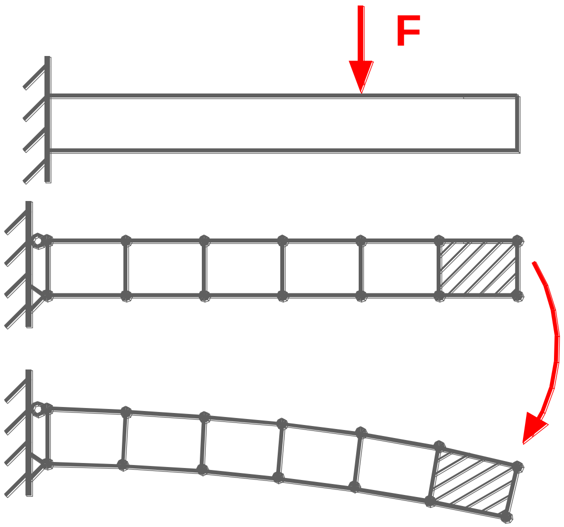
e) Rotational invariance

Finite elements for plane stress elements

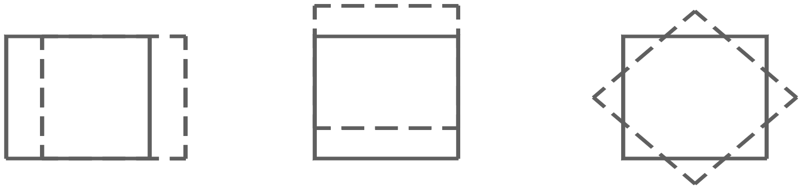
Properties of finite elements

Stress-free rigid body translation and rotation of an element

Example 1: Rigid body displacement of plane stress elements in analysis of cantilever.



Example 2: Rigid body displacement states at a plane stress element



Finite elements for plane stress elements

Properties of finite elements

Stress-free rigid body translation and rotation of an element

Example It has to be shown that for a rigid body displacement in the x-direction of the rectangular plane stress element with bilinear shape functions no nodal forces occur.

Displacement state: $u_1 = u_2 = u_3 = u_4 = 1$ $v_1 = v_2 = v_3 = v_4 = 0$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{bmatrix} = \frac{E \cdot t}{12(1-\mu^2)} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} & k_{57} & k_{58} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} & k_{67} & k_{68} \\ k_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} & k_{78} \\ k_{81} & k_{82} & k_{83} & k_{84} & k_{85} & k_{86} & k_{87} & k_{88} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{E \cdot t}{12(1-\mu^2)} \begin{bmatrix} k_{11}+k_{13}+k_{15}+k_{17} \\ k_{21}+k_{23}+k_{25}+k_{27} \\ k_{31}+k_{33}+k_{35}+k_{37} \\ k_{41}+k_{43}+k_{45}+k_{47} \\ k_{51}+k_{53}+k_{55}+k_{57} \\ k_{61}+k_{63}+k_{65}+k_{67} \\ k_{71}+k_{73}+k_{75}+k_{77} \\ k_{81}+k_{83}+k_{85}+k_{87} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

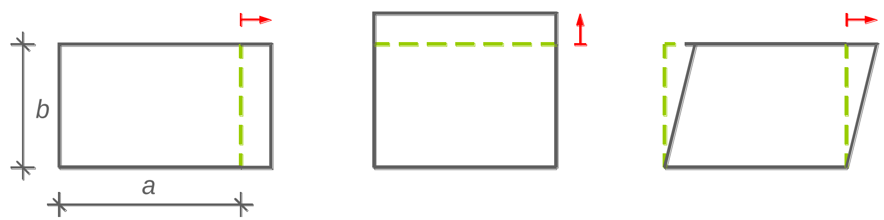
Stiffness matrix

Finite elements for plane stress elements

Properties of finite elements

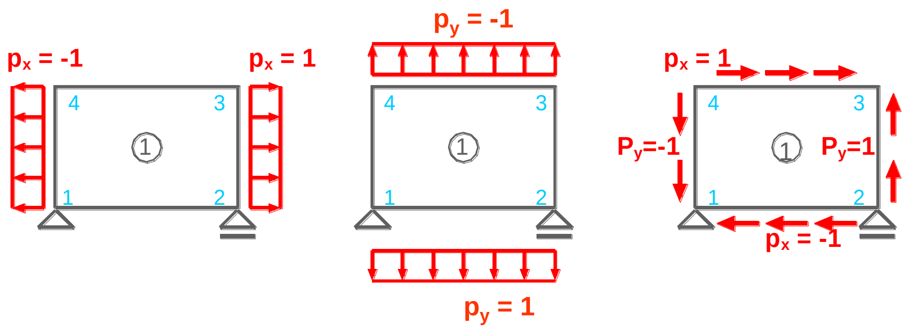
States of constant strain and constant stress

Constant strain states for a plane stress element



ε_x constant ε_y constant ε_{xy} constant

Constant stress states for a plane stress element

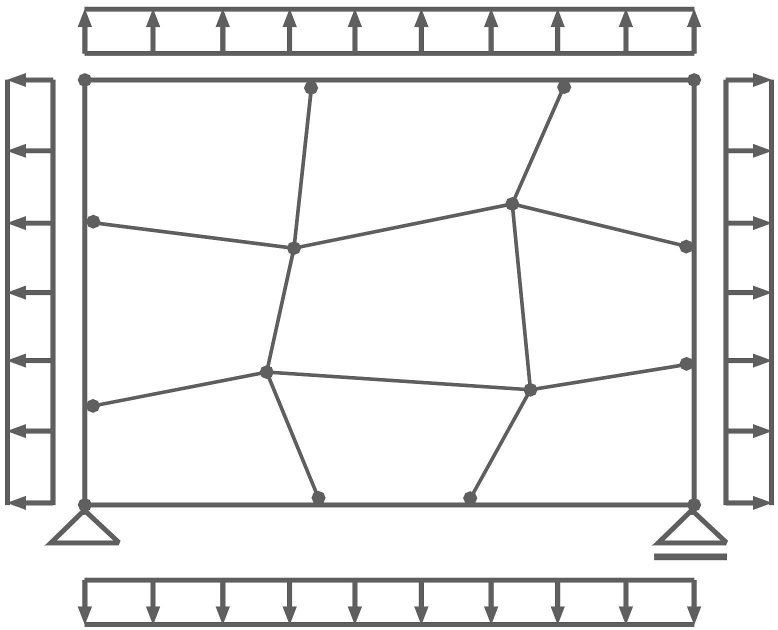


p_x, p_y [kN/m²]

Finite elements for plane stress elements

Properties of finite elements

The Patch Test



Possible element configuration

In the patch test it is checked if the „patch“ of elements is able to represent constant stress states.

Finite elements for plane stress elements

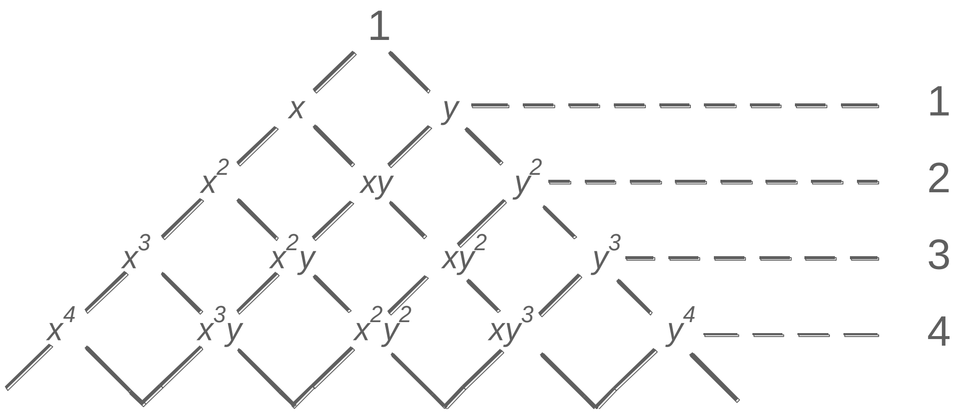
Properties of finite elements

Geometric isotropy

All elements used in practice in programs are geometrically isotropic

Rotational invariance

Requires that polynomial terms are complete



Finite elements for plane stress elements

Element types

- **Elements with continuous displacement shape functions**
  - Triangular element
  - Isoparametric elements
- **Non-conforming elements**
- **Hybrid elements**

Finite elements for plane stress elements

Displacement-based finite elements

Derivation of the element stiffness matrix for elements with displacement shape functions:

a) Determination of the shape functions of the displacements; the unknowns of the displacement functions are the nodal point displacements $\mathbf{u_e}$. (The order of the displacement function must be high enough so that all derivations needed for the calculation of the strains are not equal to zero.)

b) Determination of the strains corresponding to the displacement functions

$$\underline{\varepsilon} = \underline{B} \cdot \underline{u}_e$$

c) Formulation of the material law

$$\underline{\sigma} = \underline{D} \cdot \underline{\varepsilon}$$

d) The nodal forces corresponding to the chosen shape functions are obtained with the principal of virtual displacements as

$\underline{K}_e \cdot \underline{u}_e = \underline{F}_e$ where the element stiffness matrix is

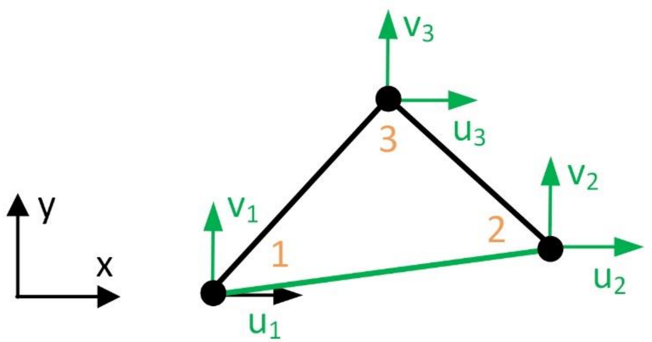
$$\underline{K}_e = \int t \cdot \underline{B}^T \cdot \underline{D} \cdot \underline{B} \, dx \, dy$$

e) Determination of the nodal loads $\underline{F}_L$ equivalent to the element loads

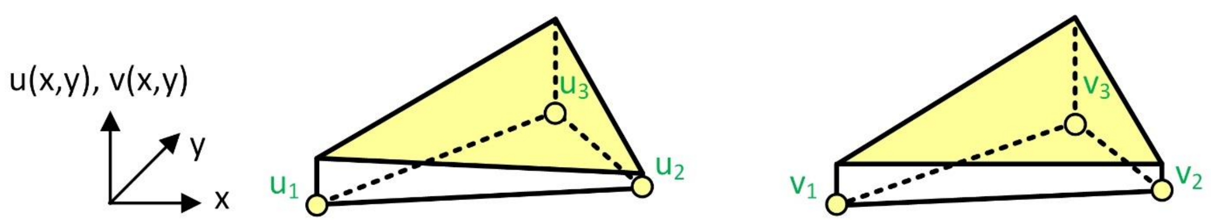
Finite elements for plane stress elements

Elements with constant displacement assumptions

Single triangular element – CST- Element (Constant Strain Triangle)



Linear displacements: $u(x,y) = α_1 + α_2 \cdot x + α_3 \cdot y$ $v(x,y) = ß_1 + ß_2 \cdot x + ß_3 \cdot y$



Strains **($ε_x, ε_y, ε_{xy}$):** constant → **Stresses ($σ_x, σ_y, σ_{xy}$): constant in the element**

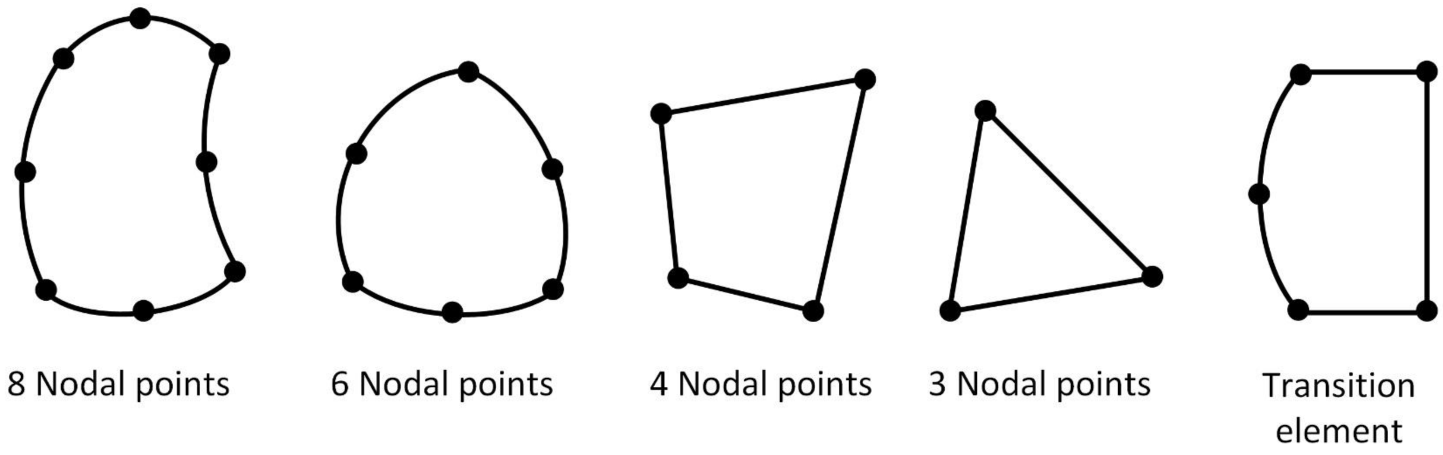
Finite elements for plane stress elements

Elements with constant displacement assumptions

Isoparametric elements

polynomial order for description of the geometry
= polynomial order of the shape functions.

Element types:

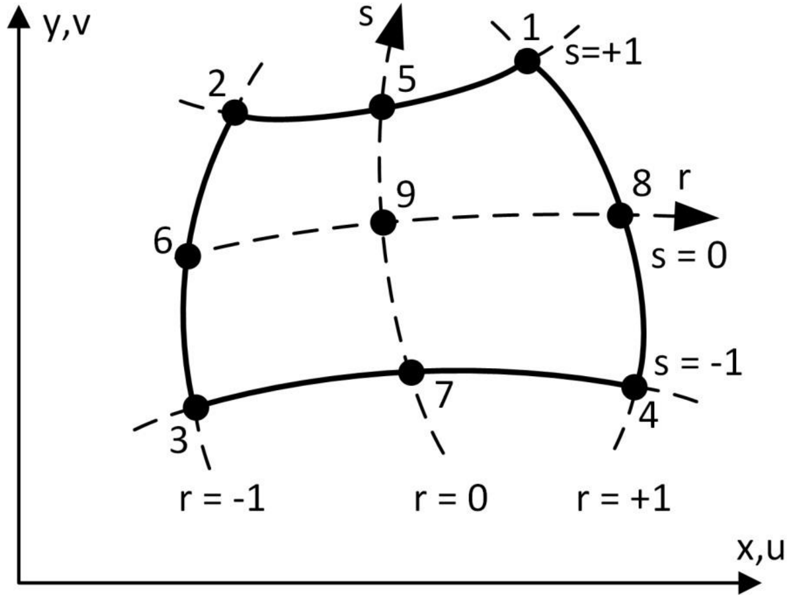


Finite elements for plane stress elements

Elements with constant displacement assumptions

Isoparametric elements

Isoparametric element with local curved coordinates r, s



Shape function:

$u = \sum h_i \cdot u_i$

$v = \sum h_i \cdot v_i$

Geometry:

$x = \sum h_i \cdot x_i$

$y = \sum h_i \cdot y_i$

Finite elements for plane stress elements

Isoparametric elements

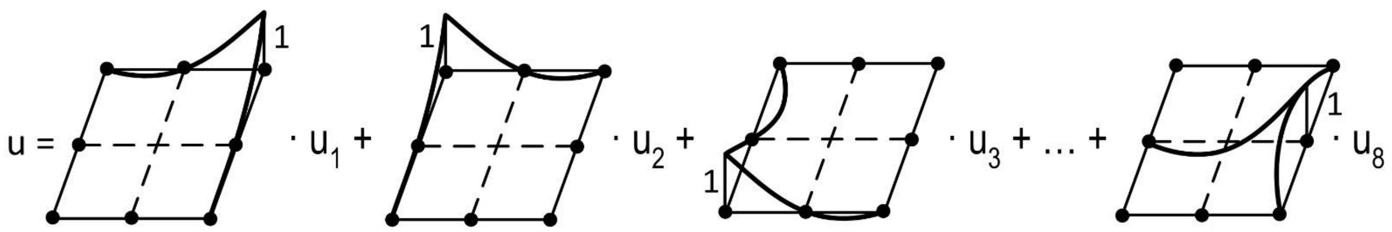
Shape functions	i = 5*	i = 6*	i = 7*	i = 8*	i = 9*
$h_1 = 1/4(1 + r)(1 + s)$	$-1/2\ h_5$	-	-	$-1/2\ h_8$	$-1/4\ h_9$
$h_2 = 1/4(1 - r)(1 + s)$	$-1/2\ h_5$	$-1/2\ h_6$	-	-	$-1/4\ h_9$
$h_3 = 1/4(1 - r)(1 - s)$	-	$-1/2\ h_6$	$-1/2\ h_7$	-	$-1/4\ h_9$
$h_4 = 1/4(1 + r)(1 - s)$	-	-	$-1/2\ h_7$	$-1/2\ h_8$	$-1/4\ h_9$
$h_5 = 1/2(1 - r^2)(1 + s)$	-	-	-	-	$-1/2\ h_9$
$h_6 = 1/2(1 - s^2)(1 - r)$	-	-	-	-	$-1/2\ h_9$
$h_7 = 1/2(1 - r^2)(1 - s)$	-	-	-	-	$-1/2\ h_9$
$h_8 = 1/2(1 - s^2)(1 + r)$	-	-	-	-	$-1/2\ h_9$
$h_9 = (1 - r^2)(1 - s^2)$	-	-	-	-	-

* if nodal point **i** is existing

Finite elements for plane stress elements

Isoparametric elements

Displacements



$$u = h_1(r,s) \cdot u_1 + h_2(r,s) \cdot u_2 + h_3(r,s) \cdot u_3 + h_4(r,s) \cdot u_4$$

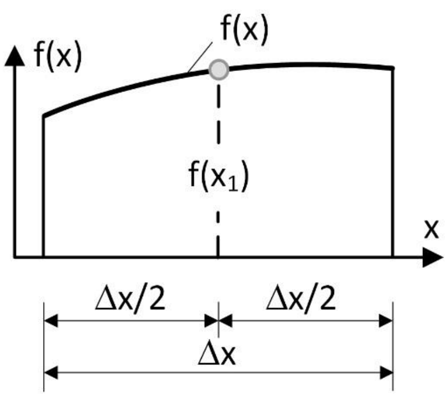
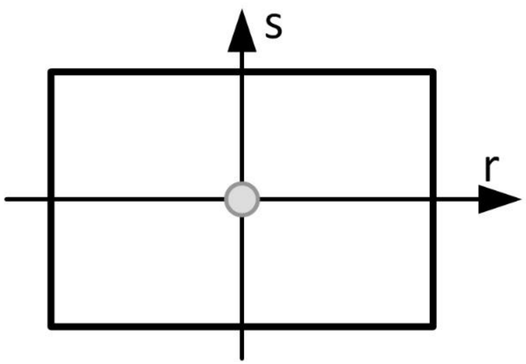
Strains $\underline{\varepsilon} = \underline{B} \cdot \underline{u}$

Stresses $\underline{\sigma} = \underline{D} \cdot \underline{\varepsilon}$

Stiffness matrix $\underline{K} = \int \underline{B}^T \cdot \underline{D} \cdot \underline{B} \, dV$ (Numerical integration according to Gauss)

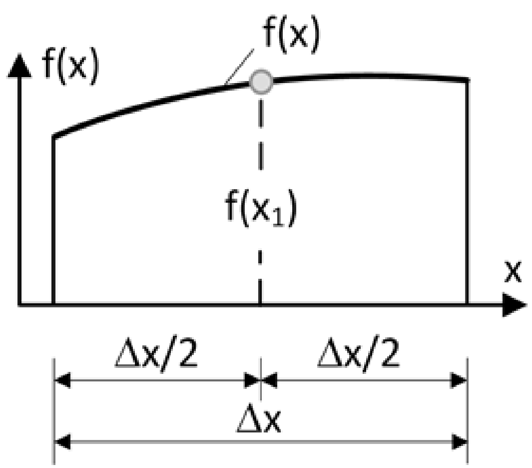
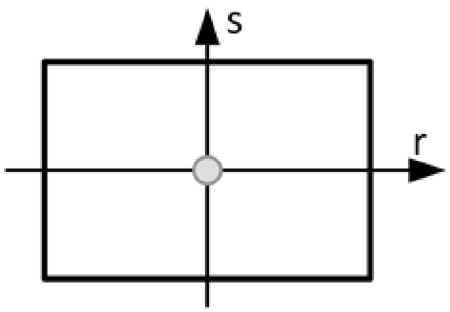
Finite elements for plane stress elements

Gaussian numerical integration

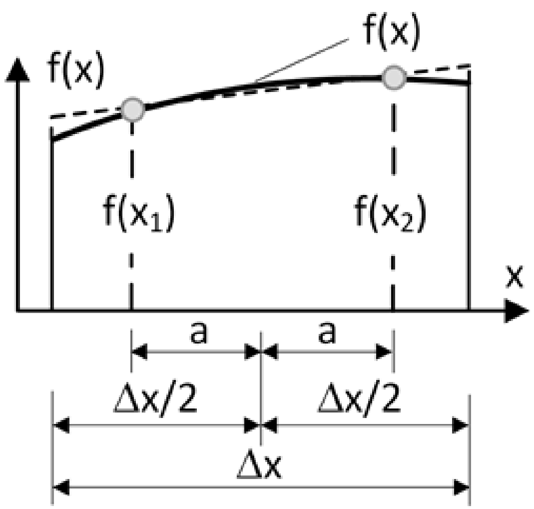
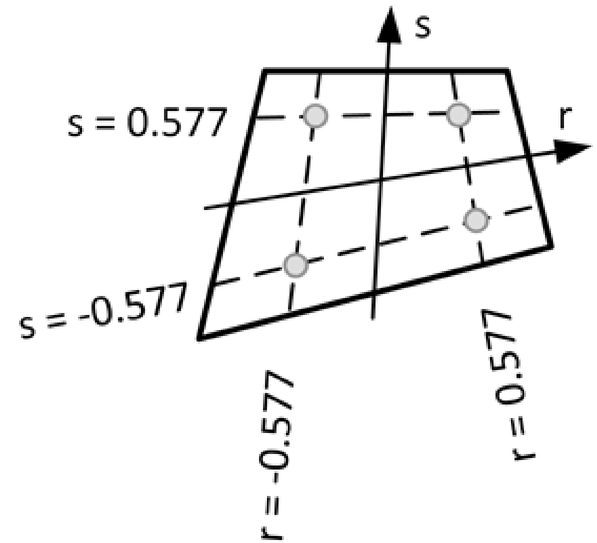
INTEGRATION ORDER	FORMULA AND ACCURACY	POSITION OF THE INTEGRATION POINTS IN FINITE ELEMENTS
	$\int_{x_a}^{x_a+\Delta x} f(x)dx = \sum_i f(x_i)\cdot\alpha_i\cdot\Delta x$	
1-point integration 	$\int_{x_a}^{x_a+\Delta x} f(x)dx = f(x_1)\cdot\Delta x$ Linear function is integrated exactly	

Finite elements for plane stress elements

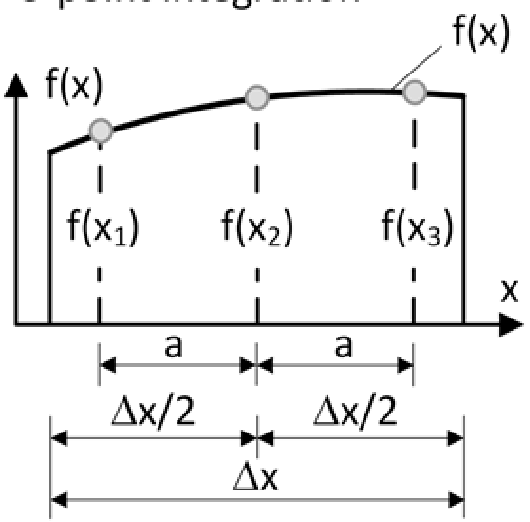
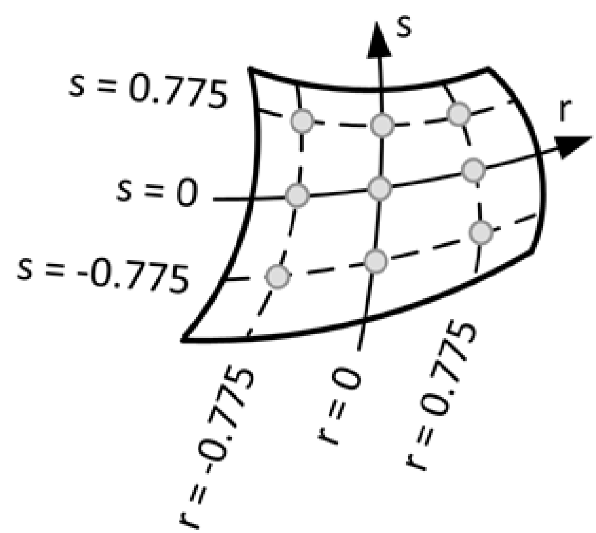
Gaussian numerical integration

Integration order n	Formula	Location of integration points $r = r_j$ and $s = s_k$ in plane finite elements
	$$\int_{x_a}^{x_a+\Delta x} f(x)\ \mathrm{d}x = \sum_i f(x_i)\cdot \alpha_i\ \frac{\Delta x}{2}$$	
1-point integration  $n=1$	$$\int_{x_a}^{x_a+\Delta x} f(x)\ \mathrm{d}x\ =\ f(x_1)\quad \alpha_1\ \frac{\Delta x}{2}$$ $$\alpha_1 = 2$$ Linear function is integrated exactly	

Finite elements for plane stress elements

Integration order n	Formula $\int_{x_a}^{x_a+\Delta x} f(x)\ \mathrm{d}x = \sum_i f(x_i) \cdot \alpha_i \dfrac{\Delta x}{2}$	Location of integration points $r = r_j$ and $s = s_k$ in plane finite elements
2-point integration  $n=2,\ a = \xi_1 \cdot \dfrac{\Delta x}{2}$	$\int_{x_a}^{x_a+\Delta x} f(x)\ \mathrm{d}x =$ $\left(\alpha_1 f(x_1) + \alpha_2 f(x_2)\right) \cdot \dfrac{\Delta x}{2}$ $\alpha_1 = 1 \qquad \alpha_2 = 1$ $\xi_1 = 1/\sqrt{3} \approx 0.577$ 3rd degree polynomial is integrated exactly	

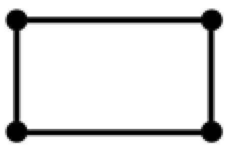
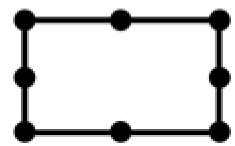
Finite elements for plane stress elements

Integration order n	Formula	Location of integration points $r = r_j$ and $s = s_k$ in plane finite elements
	$$\int_{x_a}^{x_a+\Delta x} f(x)\ \mathrm{d}x = \sum_i f(x_i) \cdot \alpha_i \ \frac{\Delta x}{2}$$	
3-point integration  $n{=}3,\ a = \xi_1 \cdot \frac{\Delta x}{2}$	$\int_{x_a}^{x_a+\Delta x} f(x)\,\mathrm{d}x = (\alpha_1 \cdot f(x_1) +$ $+\alpha_2 \cdot f(x_2) + \alpha_3 \cdot f(x_3)) \cdot \frac{\Delta x}{2}$ $\alpha_1 = \alpha_3 = 5/9 \approx 0.556$ $\alpha_2 = 8/9 \approx 0.889$ $\xi_1 = \sqrt{3/5} \approx 0.775$ 5th degree polynomial is integrated exactly	

Finite elements for plane stress elements

Elements with reduced integration

Integration order of isoparametric plane stress elements

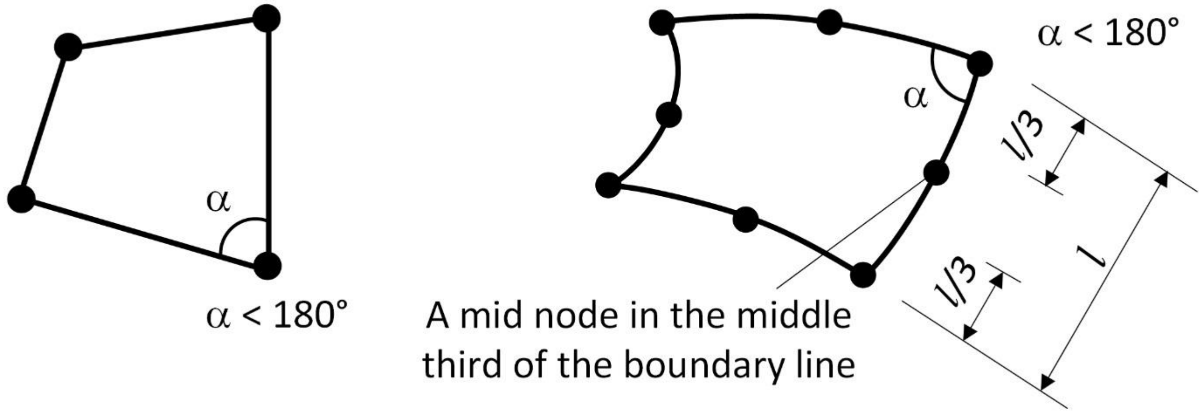
Element type	Full integration	Element type	Full integration
 4-node rectangular	2 x 2	 8-node rectangular	3 x 3
 4-node quadrilateral	3 x 3	 8-node element	4 x 4

$$\underline{K}_e = \int t \cdot \underline{B}^T \cdot \underline{D} \cdot \underline{B} \, dx \, dy$$

Finite elements for plane stress elements

Isoparametric elements

Admissible geometries of isoparametric elements

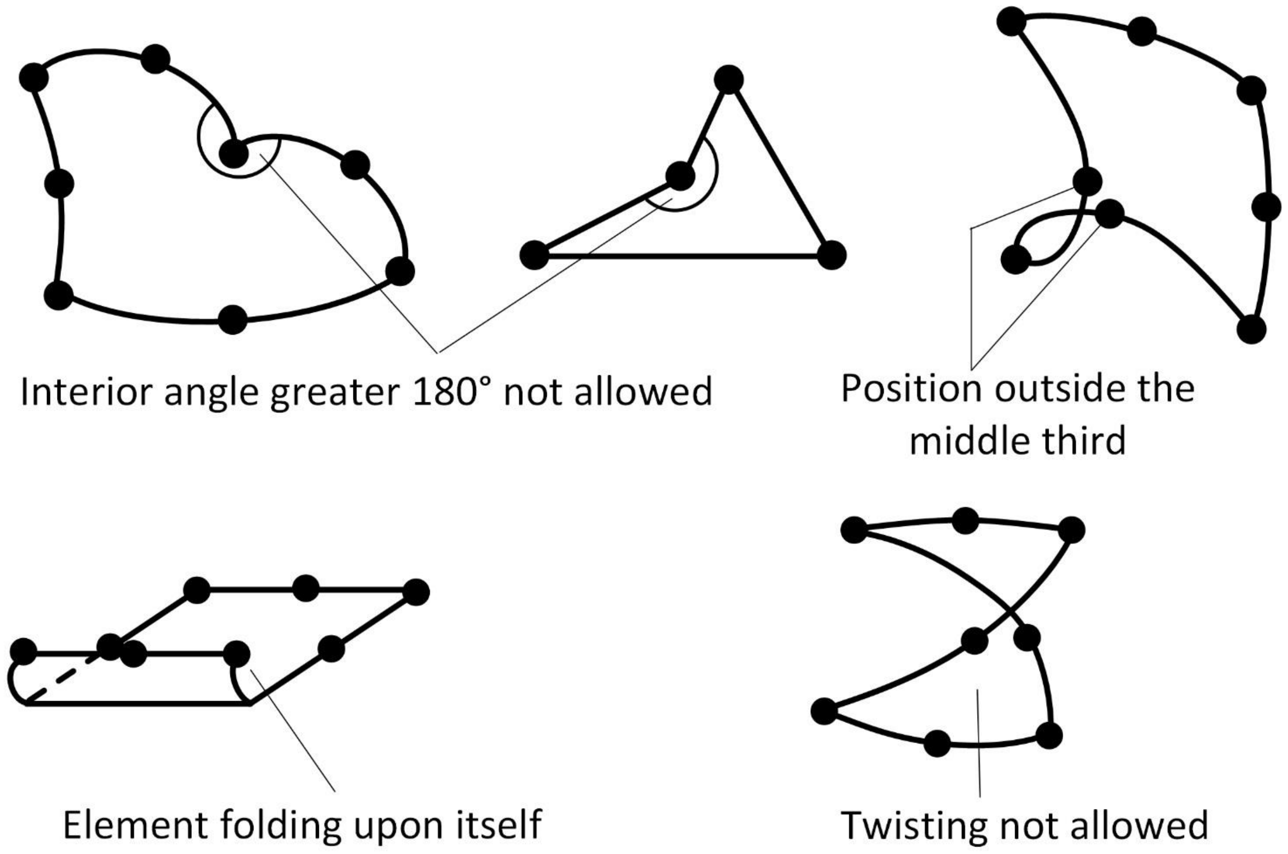


The side nodes must always lay in the middle third of the edge line!

Finite elements for plane stress elements

Isoparametric elements

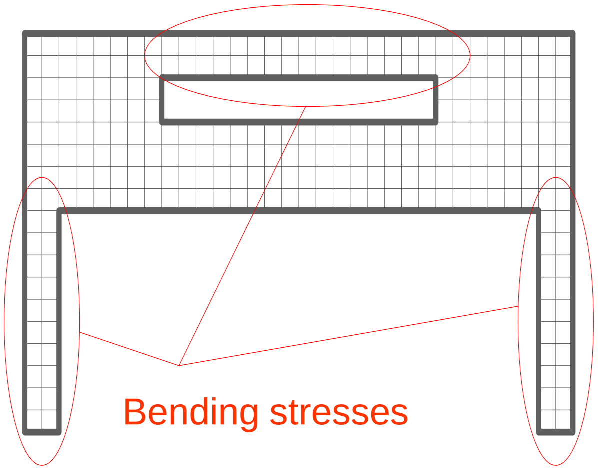
Inadmissible element geometries



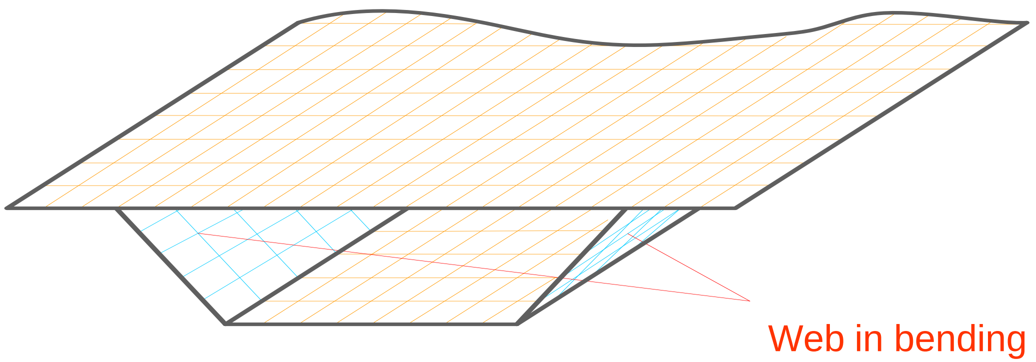
Finite elements for plane stress elements

Nonconforming Elements

Modeling of structural regions subjected to bending with plane stress elements



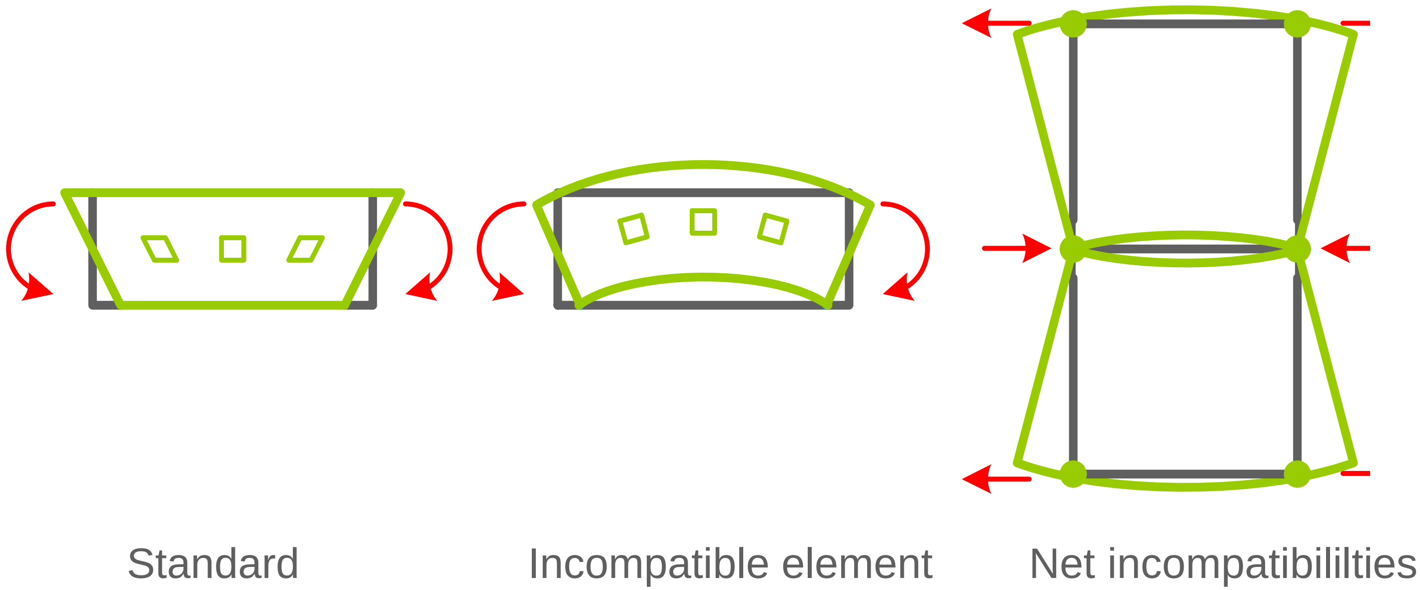
Deep beam



Box girder bridge

Finite elements for plane stress elements

Nonconforming Elements



Shape function: Extension of the bilinear shape function by quadratic terms; Elimination of the additional degree of freedom on element plane. The displacement functions at the boundaries of two adjacent elements are not compatible.

Aim: Elements able to represent „bending“ appropriately

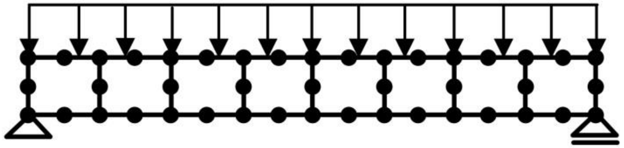
Finite elements for plane stress elements

Nonconforming Elements

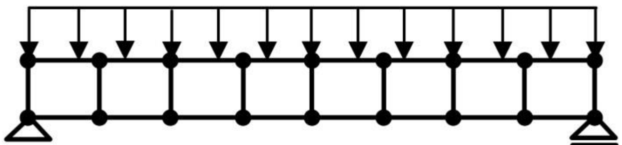
Example: Beam – analysed with plane stress elements

Convergence of stresses $\sigma_{x,m}$ [MN/m²] of a beam in bending

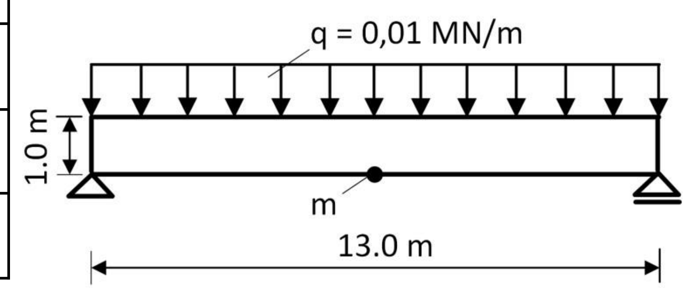
FE mesh	4-node element		8-node conforming element 2x2 underintegrated
	Conforming elements	Nonconforming elements	
2 x 1	0.06	1.27	2.96
4 x 1	0.35	2.22	2.64
8 x 1	1.06	2.45	2.56
16 x 1	1.90	2.51	2.54
32 x 1	2.34	2.53	-
64 x 1	2.48	-	-
128 x 1	2.52	-	-



8 x 1 elements – 8-node element



8 x 1 elements – 4-node element



$$M_m = \frac{q \cdot l^2}{8} = \frac{0.01 \cdot 13^2}{8} = 0.211 \text{ MNm}, \quad W = \frac{t \cdot h^2}{6} = 0.083 \text{ m}^3 \qquad \sigma_{x,m} = \frac{M_m}{W} = \frac{0.211}{0.083} = 2.535 \text{ MN/m}^2$$

Finite elements for plane stress elements

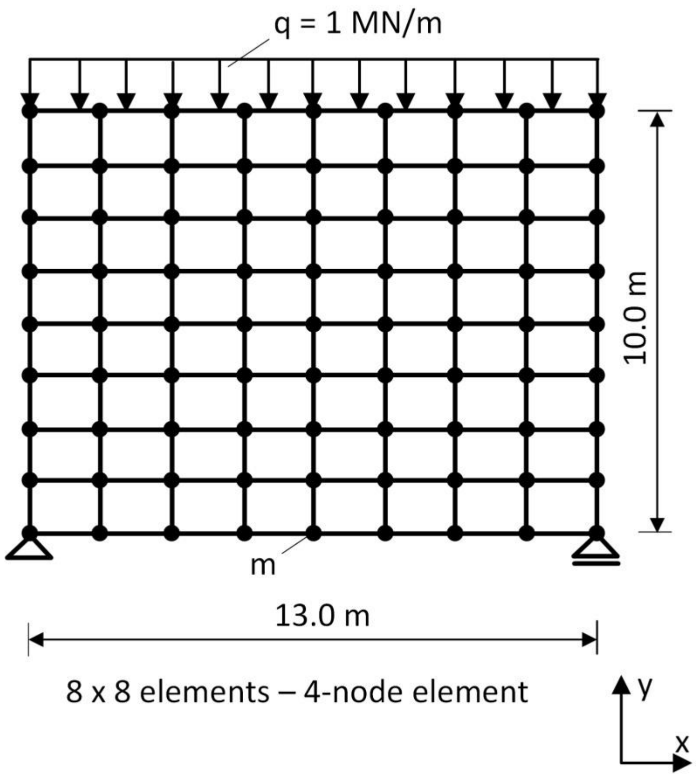
Nonconforming Elements

Example: Plate – analysed with plane stress elements

Convergence of the stresses $\sigma_{x,m}$ [MN/m²] of a deep beam

FE mesh	Conforming elements	Nonconforming elements
2 x 2	1.66	2.28
4 x 4	4.32	4.72
8 x 8	4.22	4.22

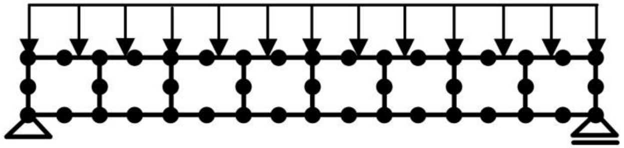
Convergence of stresses $\sigma_{x,m}$ [MN/m²] of a beam in bending



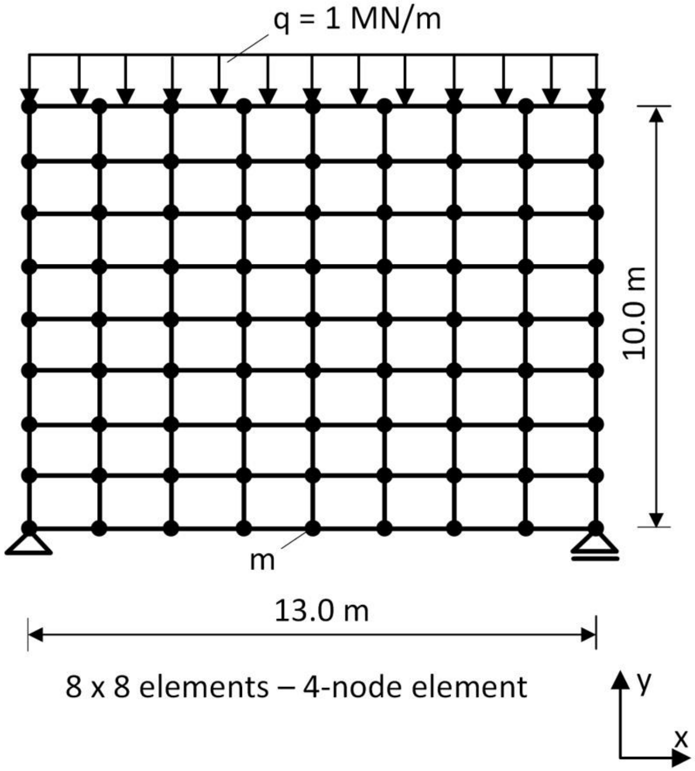
Finite elements for plane stress elements

Nonconforming Elements

Example: Plate – analysed with plane stress elements



8 x 1 elements – 8-node element



8 x 8 elements – 4-node element

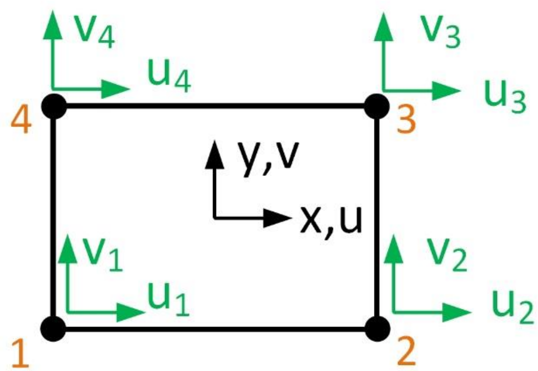
Conclusions

- The 4-node plane stress element is not well suited to model structural parts of plates subjected to bending (shear locking).
- Appropriate elements to model bending are
  - nonconforming 4-node elements
  - isoparametric elements with higher shape functions.
  - hybrid elements

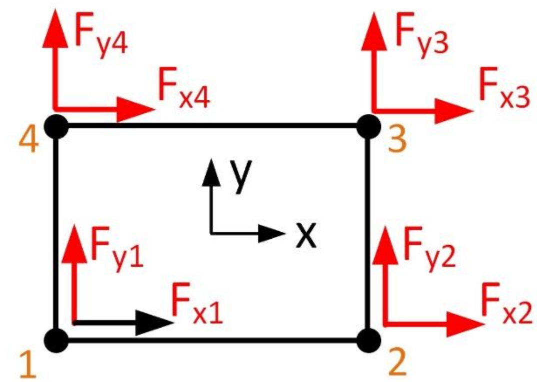
Finite elements for plane stress elements

Hybrid Elements

Hybrid Plate Element



Displacements

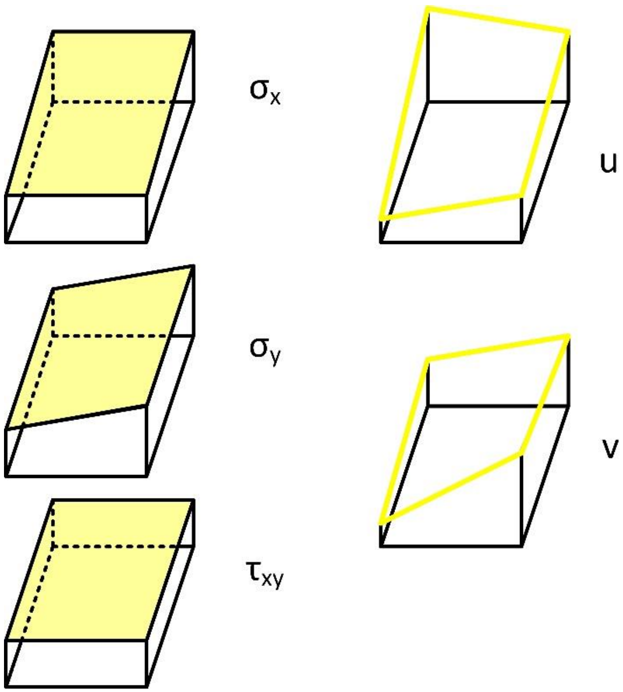


Forces

The basic assumptions for hybrid elements are shape functions for the stress inside the element and shape functions for the displacements at the element boundaries.

Finite elements for plane stress elements

Hybrid Elements



SHAPE FUNCTIONS FOR STRESSES INSIDE THE ELEMENT

SHAPE FUNCTIONS FOR DISPLACEMENTS AT THE ELEMENT EDGES

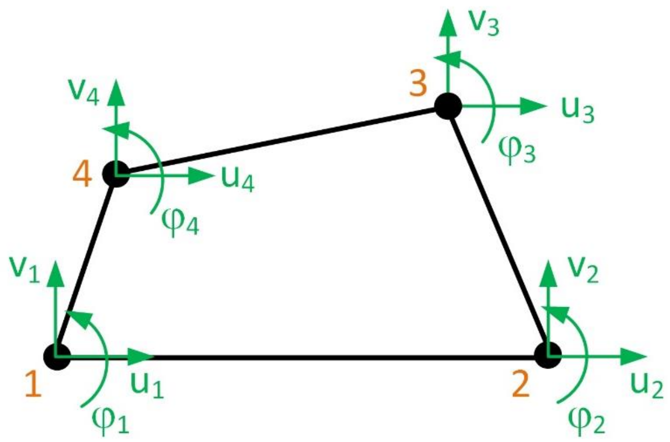
Principles of virtual work

The stress shape functions result in strains inside the element. The displacements corresponding to those strains are „adjusted" at the displacement shape functions at the boundaries by the principle of virtual work. The principle of virtual displacements leads to an element stiffness matrix with displacements as unknowns as in the case of displacement-based elements.

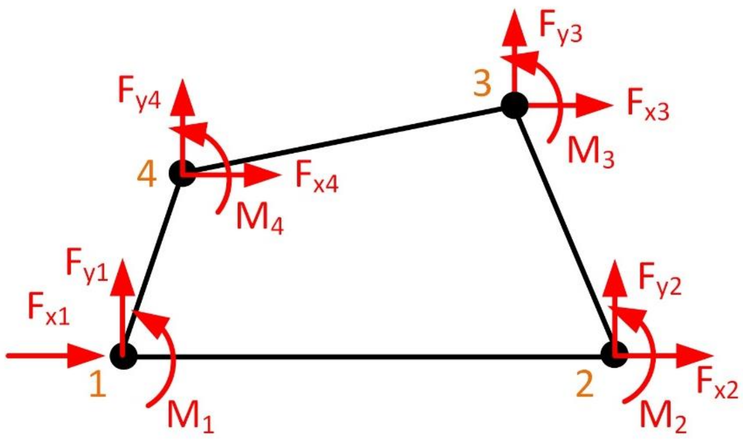
Finite elements for plane stress elements

Hybrid Elements

Hybrid plane stress element with rotational degrees of freedom



Displacements



Forces

End

Introduction

Truss and beam structures

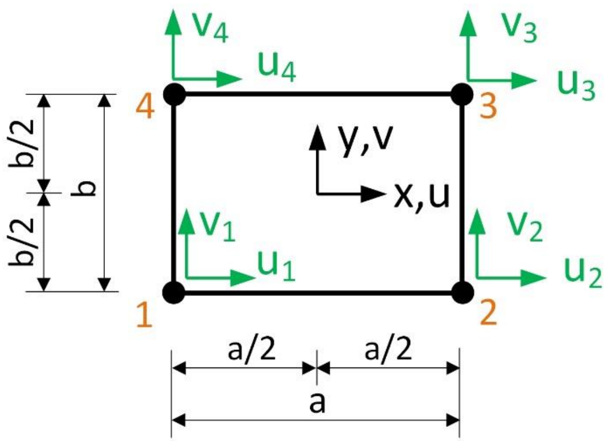
Plate and shell structures

Modeling

Rectangular element for plates in plane stress

Stiffness matrix for the rectangular element for plates in plane stress

with



$$k_{11} = k_{33} = k_{55} = k_{77} = 4\,b/a + 2\,(1-\mu)\,a/b$$
$$k_{22} = k_{44} = k_{66} = k_{88} = 4\,a/b + 2\,(1-\mu)\,b/a$$
$$k_{12} = k_{47} = k_{38} = k_{56} = 3/2\,(1+\mu)$$
$$k_{13} = k_{57} = -4\,b/a + (1-\mu)\,a/b$$
$$k_{14} = k_{27} = k_{58} = k_{36} = -3/2\,(1-3\,\mu)$$
$$k_{15} = k_{37} = -2\,b/a - (1-\mu)\,a/b$$
$$k_{16} = k_{25} = k_{78} = k_{34} = -3/2\,(1+\mu)$$
$$k_{17} = k_{35} = 2\,b/a - 2\,(1-\mu)\,a/b$$
$$k_{18} = k_{23} = k_{67} = k_{45} = 3/2\,(1-3\,\mu)$$
$$k_{24} = k_{68} = 2\,a/b - 2\,(1-\mu)\,b/a$$
$$k_{26} = k_{48} = -2\,a/b - (1-\mu)\,b/a$$
$$k_{28} = k_{46} = -4\,a/b + (1-\mu)\,b/a$$