Finite Elements in Structural Analysis

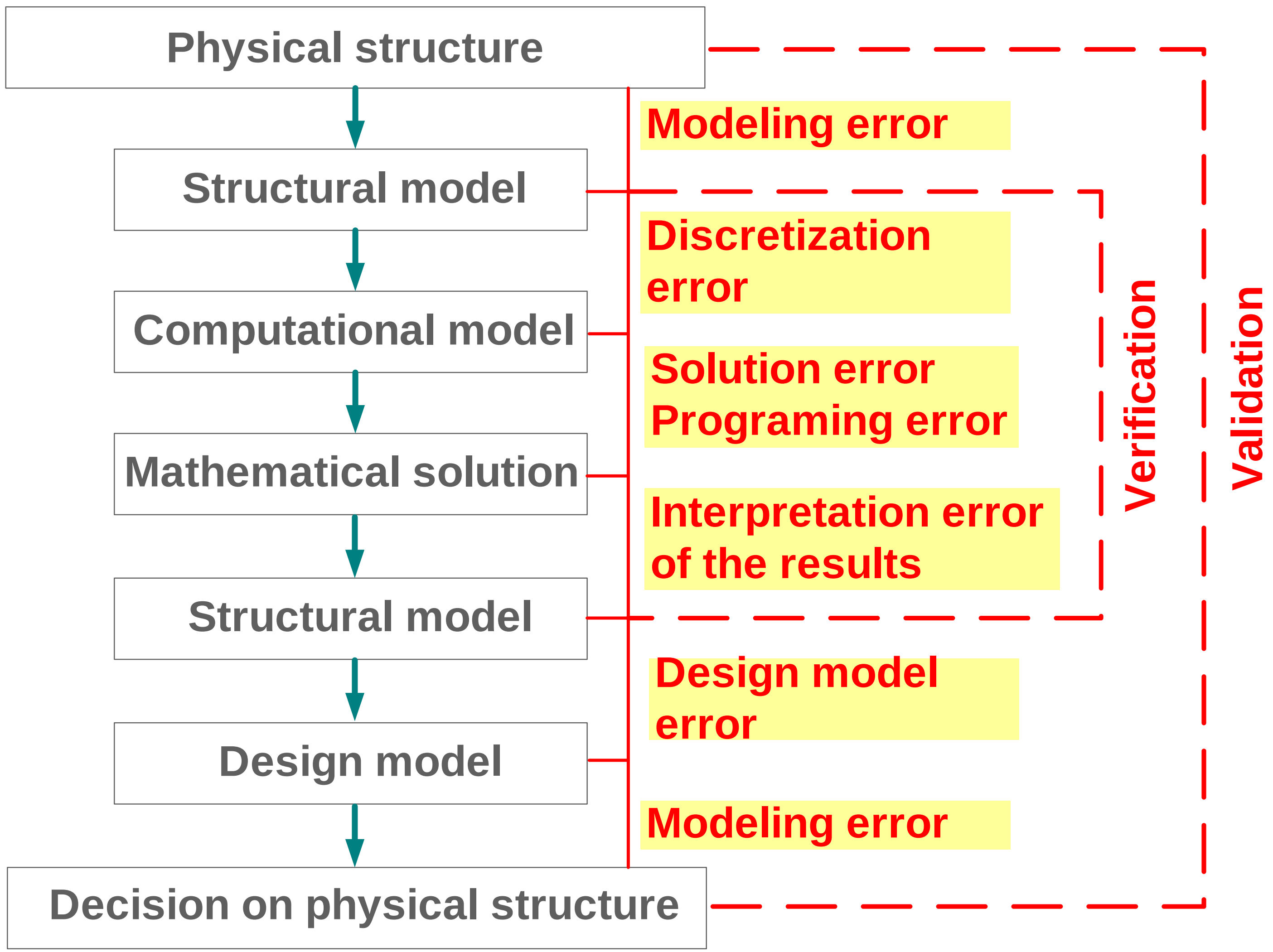
Introduction

Truss and beam structures

Plate and shell structures

4 Modeling and quality assurance

Structural models

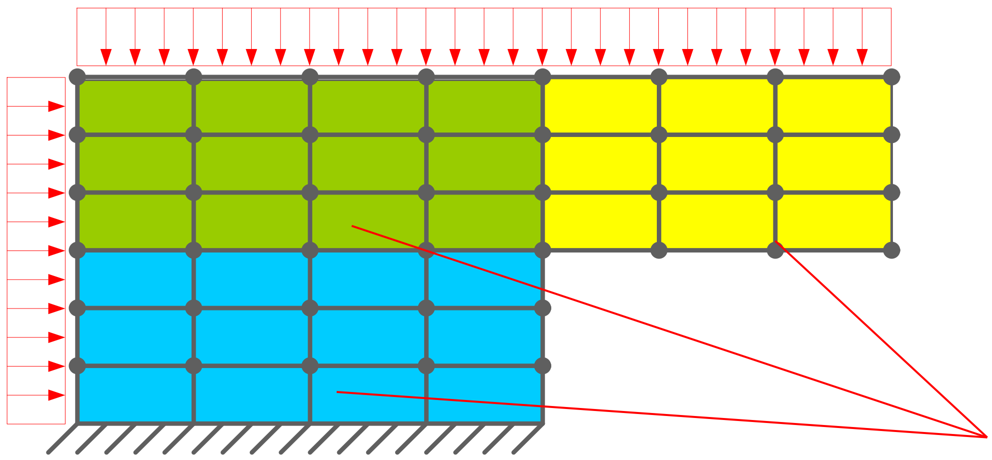


Some topics in FE modeling

Content

- Element discretization
- Singularities
- Some pitfalls in structural modeling
- Interpretation of the results
- Quality assurance of FE results

Rules for the modeling of plate regions



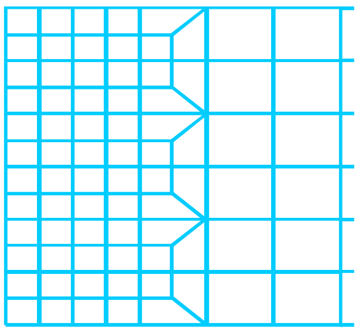
- Nets should be regular
- Element dimensions: approx. 8 -12 elements on the shorter side of a plate region
- Stress singularities at corners
- Plate parts subjected to bending should be modeled suitably

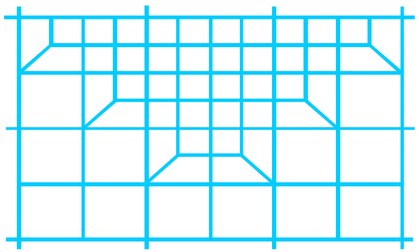
Element types and meshing

Rules for finite element meshes

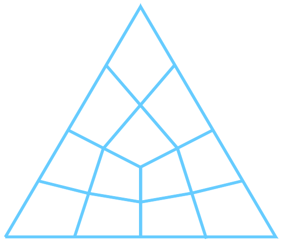
- For numerical accuracy, best element shapes are square or rectangular.
- Quadrilateral elements should be preferred to triangular elements.
- A mesh consisting purely of quadrilateral elements has to be preferred to a mixed element topology consisting of triangles and quadrilaterals.
- For a constant accuracy, element meshes have to be refined in areas with a high stress gradient.
- Size changes of elements should be smooth to avoid “artificial stiffness jumps” due to meshing.
- Stiffness “jumps” due to changes of the plate thicknesses are not allowed to be arbitrarily large.

Examples: meshing with quadrilateral elements

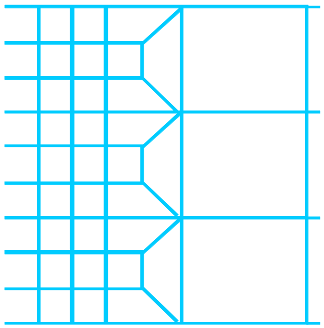




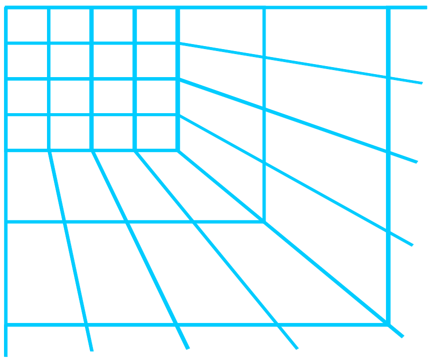
Mesh refinement at a point



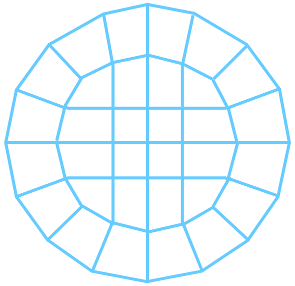
Triangular



Mesh refinement in one direction



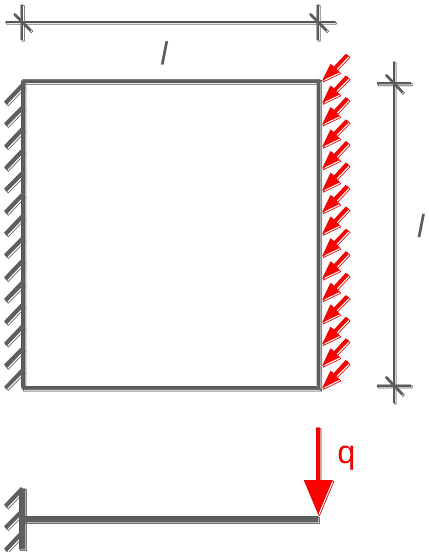
transition to halfspace



Circle

Example: regularity of FE meshes

Cantilever plate with a line load



Comparison of 3 FE meshes and two element types

- Hybrid plate element
  (Kirchhoff plate theory)

- Deformation based plate element
  (Mindlin plate theory)

l =10, *q* =5, *d* / *l* =0.1

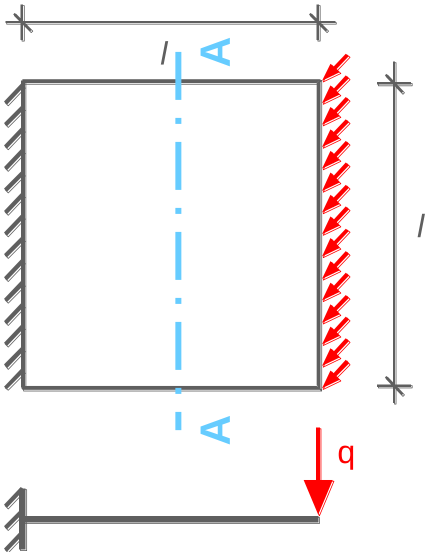
t = taper

d= plate thickness

Element type and meshing

Example: regularity of FE meshes

Cantilever plate with a line load



Exact solution:

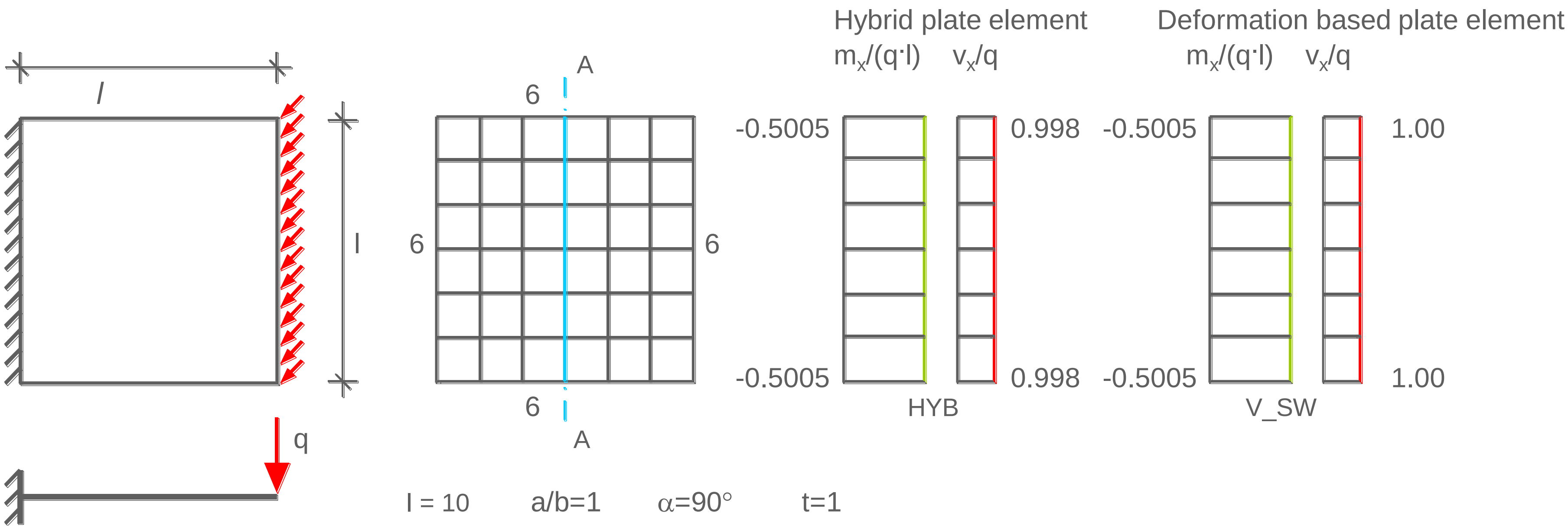
$$v_x = q \rightarrow v_x/q = 1.00$$

$$m_x = -q \cdot \frac{\ell}{2} \rightarrow m_x/(q \cdot \ell) = -0.500$$

Element type and meshing

Example: regularity of FE meshes

FE-mesh 1: regular mesh



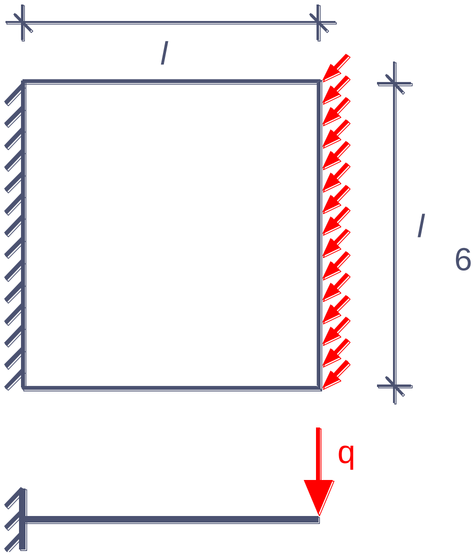
t / *l* = 0.1
Shape factors: a/b=1 α=90°

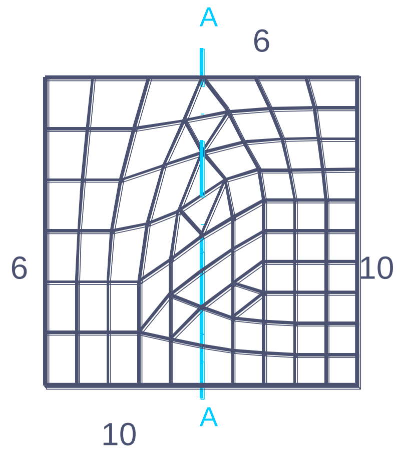
FE internal forces (section A-A)

Element type and meshing

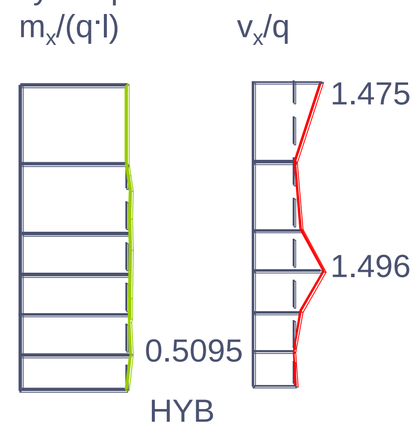
Example: regularity of FE meshes

FE-mesh 2: irregular finite element mesh with mixed element topology

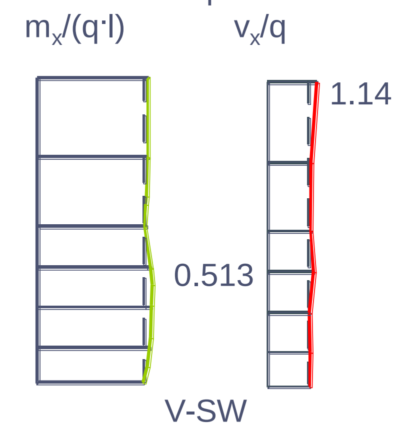




Hybrid plate element



Deformation based plate element

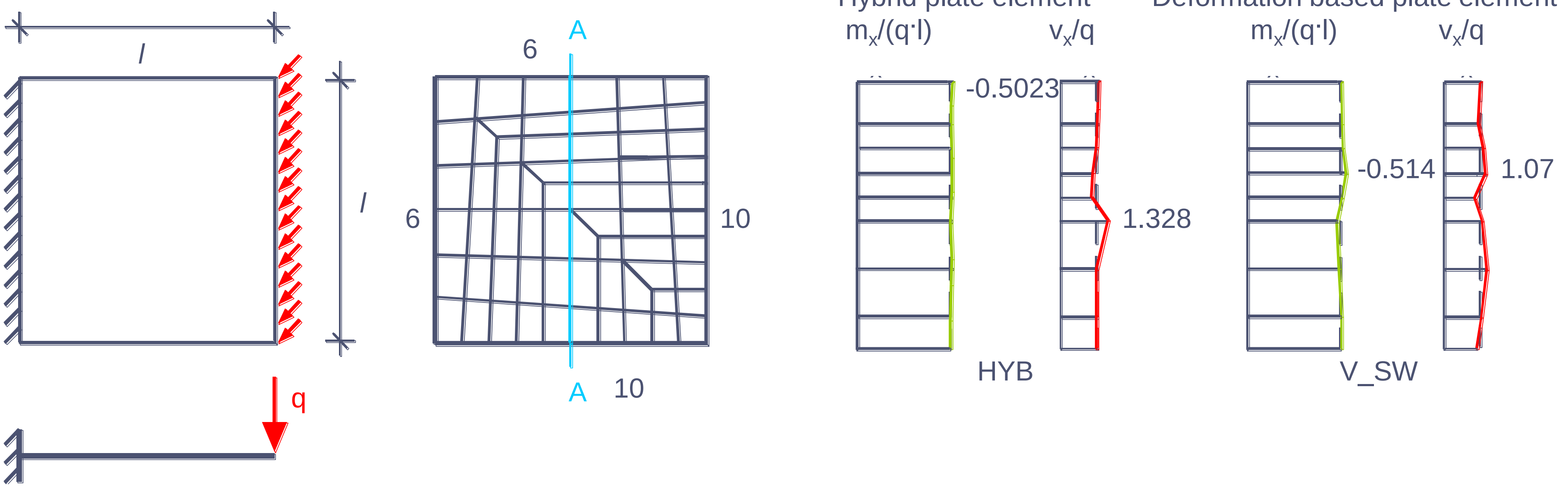


t / *l* = 0
Shape factors: a/b=1.8 α=35°

FE internal forces (section A-A)

Example: regularity of FE meshes

FE-mesh 3: irregular finite element mesh with rectangular elements



t / *l* = 0.063
Shape factors : a/b=1.9 α=59°

FE internal forces (section A-A)

Element type and meshing

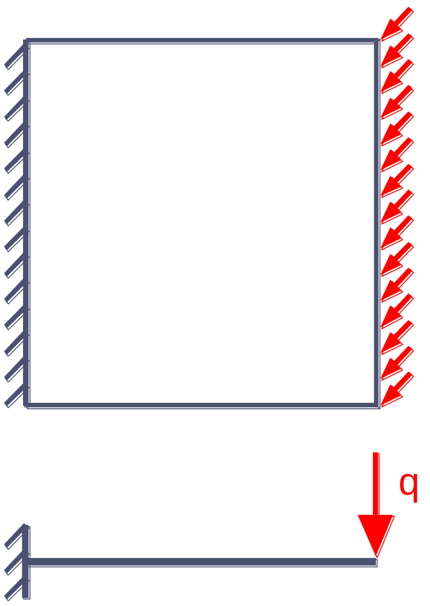
Example: regularity of FE meshes

Influence of slab thickness on numerical accurity of shear forces

ELEMENT TYPE	slab thickness d	v_x/q - FE-mesh 2	v_x /q - FE-mesh 3
exact	*for all d*	1.00	1.00
hybrid plate element	*for all d*	1.50 (50%)	1.33 (33%)
deformation based plate element	$0.02 \cdot \ell$	2.15 (115%)	1.52 (52%)
	$0.05 \cdot \ell$	1.45 (45%)	0.76 (24%)
	$0.10 \cdot \ell$	1.14 (14%)	0.85 (15%)
	$0.30 \cdot \ell$	1.03 (3%)	1.03 (3%)

Element type and meshing

Example: regularity of FE meshes

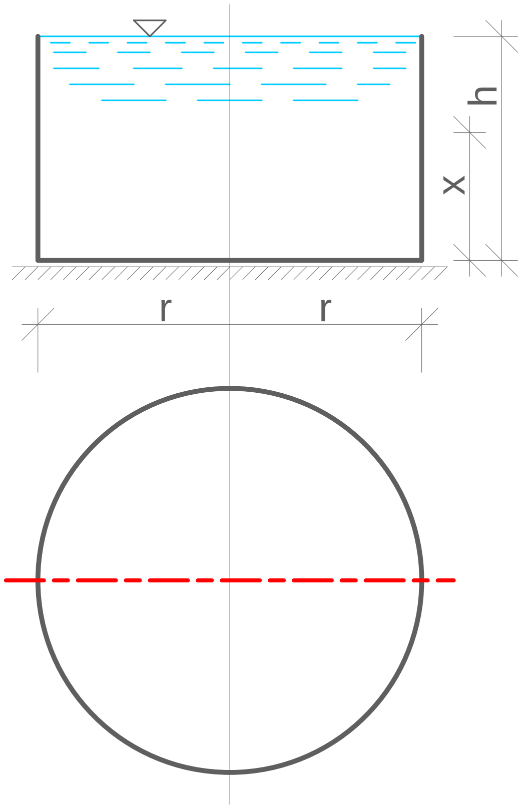


Results

- A rectangle is the numerically most insensitive element shape.
- Distorted elements may result in large errors in the internal forces. This applies especially for numerically sensitive values, e.g. shear forces of slabs.
- For plate elements with shear deformations the accuracy of shear forces increases with an increasing plate thickness.

Example: Cylindrical shell with water pressure

Analytical solution



Cylindrical tank with constant wall thickness

Ring tension force

$$N_\varphi = \gamma \cdot r \cdot \left[h - x - h \cdot e^{-\kappa \cdot x/r} \cdot \cos\frac{\kappa \cdot x}{r} + \left(\frac{r}{\kappa} - h\right) \cdot e^{-\kappa \cdot x/r} \cdot \sin\frac{\kappa \cdot x}{r} \right]$$

Bending moment

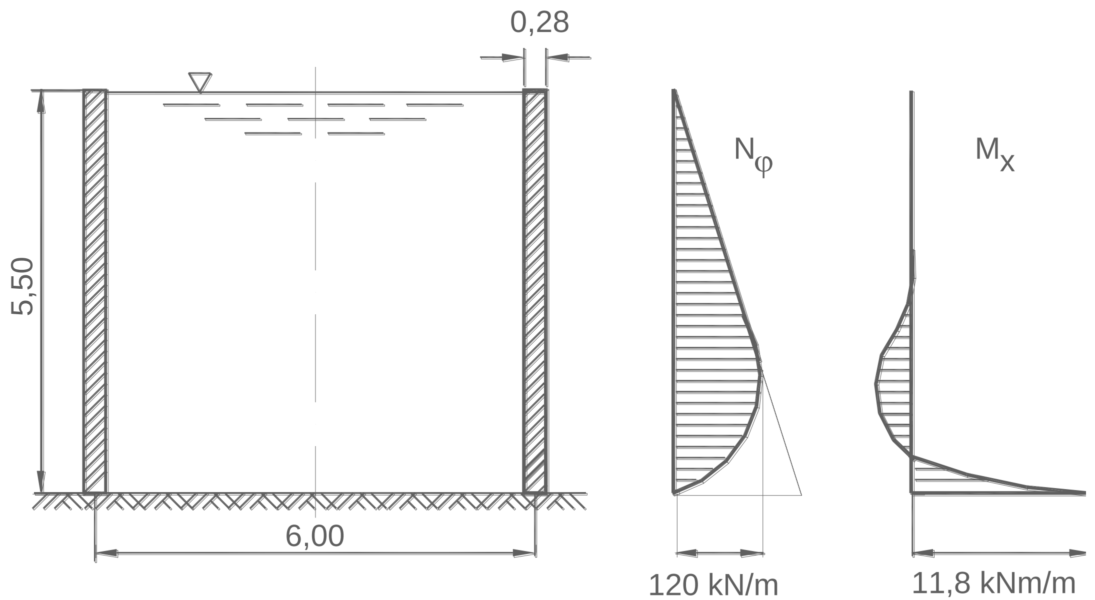
$$M_x = -\frac{\gamma \cdot r \cdot t}{\sqrt{12 \cdot (1-\mu^2)}} \cdot \left[\left(\frac{r}{\kappa} - h\right) \cdot e^{-\kappa \cdot x/r} \cdot \cos\frac{\kappa \cdot x}{r} + h \cdot e^{-\kappa \cdot x/r} \cdot \sin\frac{\kappa \cdot x}{r} \right]$$

$$\kappa = \sqrt[4]{3 \cdot \frac{r^2}{t^2}(1-\mu^2)}$$

Example: Cylindrical shell with water pressure

Sectional forces by FEM and analytical solution

Parameters: h= 5.5 m, r = 3.0 m, t = 0.28 m, μ= 0.2, x=0



System

analytical sectional forces acc. to Flügge

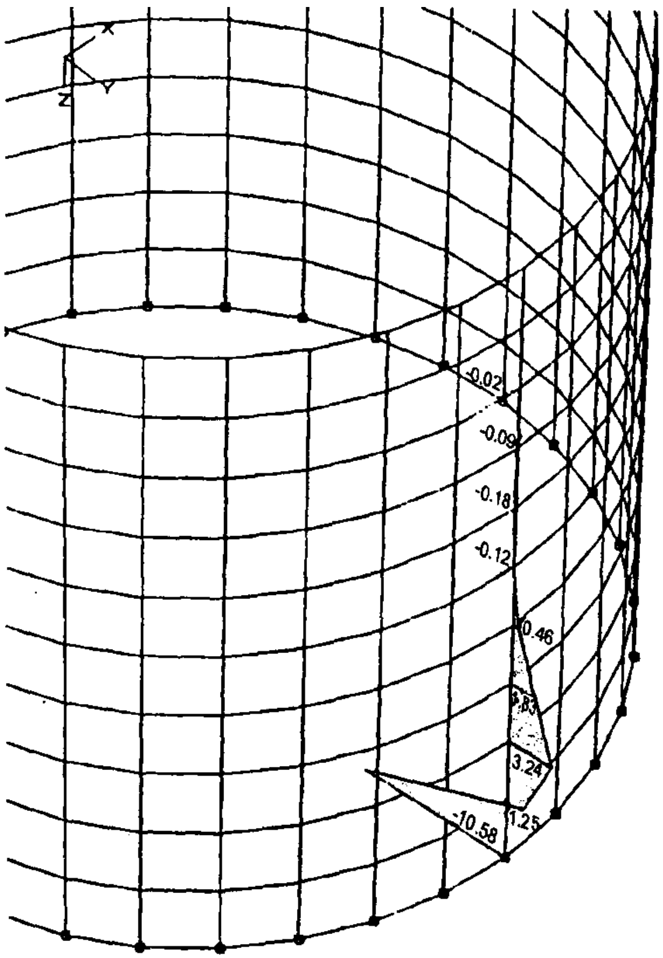
Restraining moments in shells

- peak values are abtained at the restraints.
- restraining moments rapidly decay, since membrane action prevails.
- Elastic restraints instead of rigid restraints reduces the peaks of the sectional forces considerably.
- A boundary layer with twidth ℓ_0 must be discretizised with finer elements.

Width of the boundary layer:

$$\ell_0 = \sqrt{t \cdot r}$$

r : radius of the cylinder
t : wall thickness



Boundary layer:

$$\ell_0 = \sqrt{t \cdot r} = \sqrt{0.28 \cdot 3.0} = 0.917\,\text{m}$$

Element size:

$$e = h/10 = 5.5/10 = 0.55\ \text{m}$$

$$l_0/e = 0.91/0.55 = 1.67$$

→ Boundary layer requires a finer meshing!

Cylindrical shell
Finite element modell with $e = h/10$

	Element size [m]	$\frac{\ell_0}{e}$	m_y [kNm/m]	(%Fehler)	v_y [kN/m]	n_x [kN/m]
analytical	**-**	**-**	**11.87**	**(0%)**	**36.2**	**120.0**
FEM	0.917	1	8.71	(27%)	13.9	123.0
	0.550	1.67	10.58	(11%)	21.5	119.9
	0.458	2	10.95	(8%)	23.8	116.9
	0.229	4	11.60	(2%)	29.8	117.0

Sectional forces of the shell

Result

- A minimum number of 4 elements is required in the boundary layer of width ℓ_0 , if the boundary is fixed. In the case of an elastically restrained boundary, less elements may be sufficient.

Definition

In mathematics, a singularity is in general a *point* at which a given *mathematical object is not defined*, or a point of an exceptional set where it fails to be well-behaved in some particular way, such as differentiability. (Wikipedia)

Singularities are often also called singular points.

Journal of Singularities

Example

Example: Point force

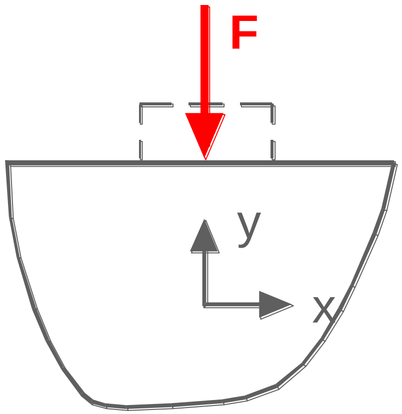
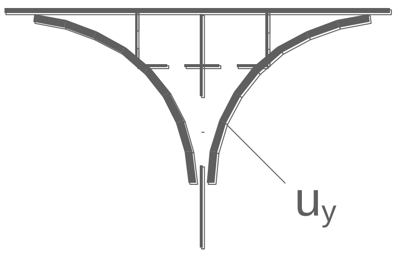


Plate in plane stress

Stress $\sigma_y = \frac{F}{A} \Rightarrow \lim_{A \to 0} \sigma_y = \infty$

Displacement $\lim_{A \to 0} u_y = \infty$



Singularities of stresses and displacements indicate a deficiency of the structural model and not of FEM!

Example

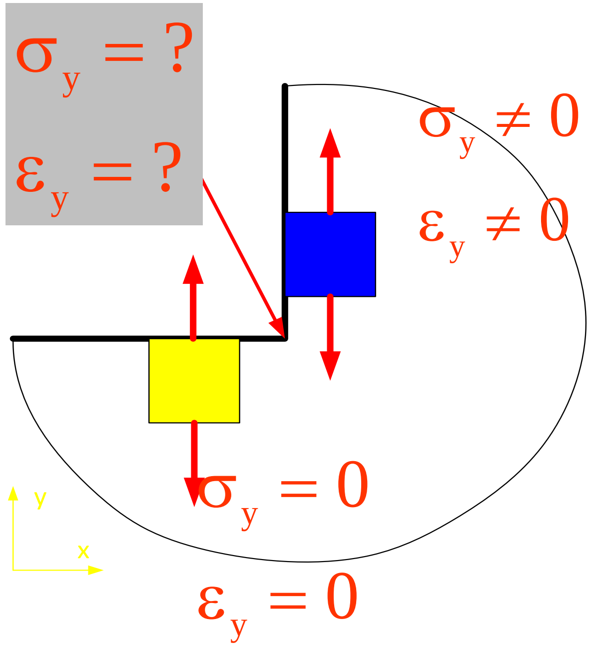
Example: Reentrant corner

The problem

Stresses and strains are not defined in the corner point!

Improved model

Round corners avoid the singularity

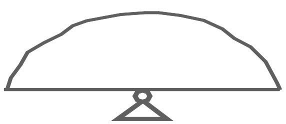
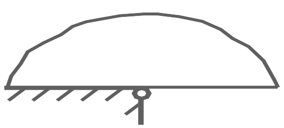


Modeling of plates in plane stress

Supports

- Point supports → singularity
- Rigid line supports → singularity
- Elastic supports → converging results

Singularities at supports of plates

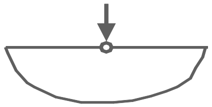
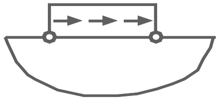
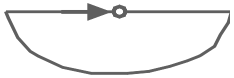
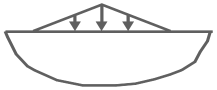
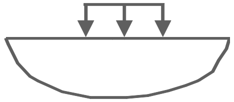
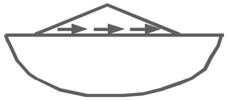
Support	Stresses	Displacements
	yes	yes
	yes	no

Modeling of plates in plane stress

Loads

- Point loads result in singularities
- Point loads may be defined more realistically as distributed loads

Singularities of loads

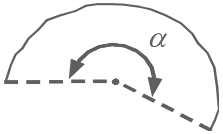
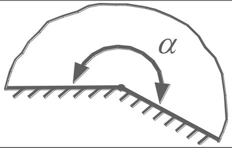
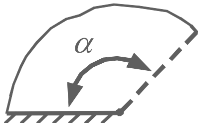
Load	Stresses	Displacements	Load	Stresses	Displacements
	yes	yes		yes (σ_x)	no
	yes	yes		no	no
	no	no		no	no

Modeling of plates in plane stress

Plate regions

Singularities of stresses

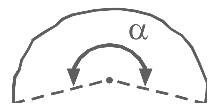
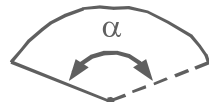
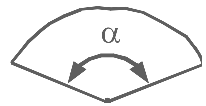
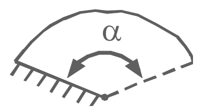
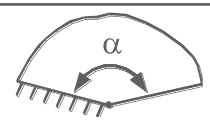
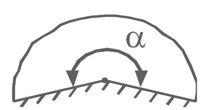
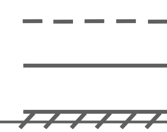
(plane stress)

Support	Stresses singular for
	$\alpha > 180^0$
	$\alpha > 180^0$
	$\alpha > 63^0$
Type of support	free  fixed

Modeling of plates in bending

Singularities of sectional forces at corner points

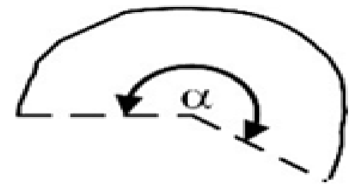
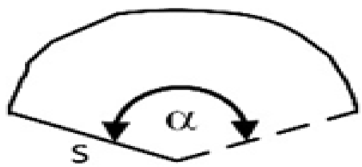
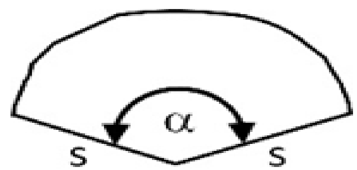
- Kirchhoff plate -

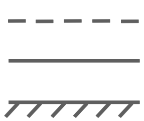
Support conditions	moments	shear forces
	$\alpha > 180°$	$\alpha > 78°$
	$\alpha > 90°$	$\alpha > 51°$
	$\alpha > 90°$	$\alpha > 60°$
	$\alpha > 95°$	$\alpha > 52°$
	$\alpha > 129°$	$\alpha > 90°$
	$\alpha > 180°$	$\alpha > 126°$
support conditions 	free simply supported fixed	

Modeling of plates in bending

Singularities of sectional forces at corner points

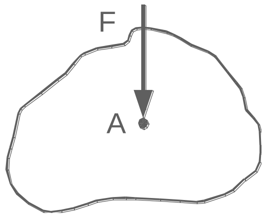
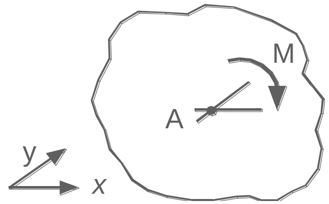
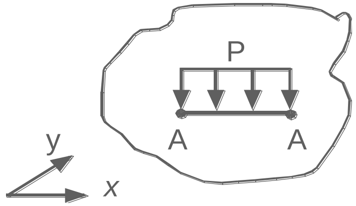
- Mindlin-Reissner plate -

Support conditions	Bending moments → ∞ for	Shear forces → ∞ for
	$\alpha > 180°$	$\alpha > 180°$
	$\alpha > 180°$	$\alpha > 90°$
	$\alpha > 180°$	$\alpha > 180°$

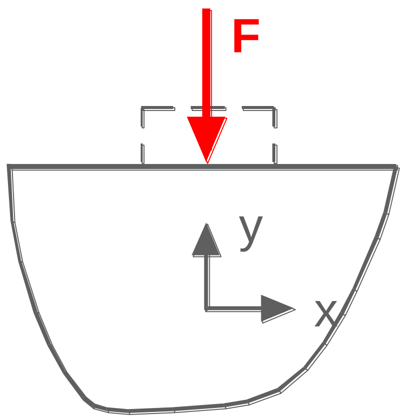
support conditions 
free
simply supported (s=soft support)
fixed

Modeling of plates in bending

Singularities at loading points – Kirchhoff plate

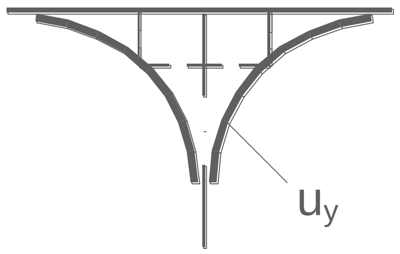
Load	Displacements	Sectional forces
	**no**	**yes** (point A: m_x, m_y, q_x, q_y)
	**yes** - point A	**yes** (point A: m_{xy}, q_y)
	**no**	yes (point A: q_x)

Dealing with singularities



Example: Point force

Stress $$\sigma_y = \frac{F}{A} \Rightarrow \lim_{A \to 0} \sigma_y = \infty$$



In Finite Element Analysis

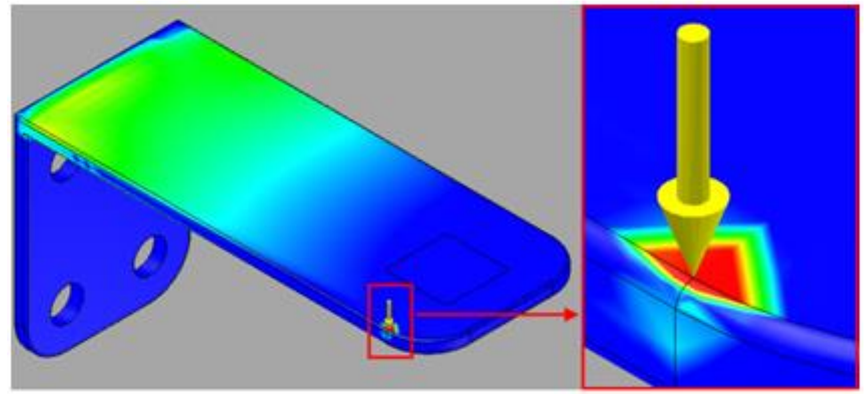
STRESS SINGULARITIES = HIGH LOCALIZED STRESSES

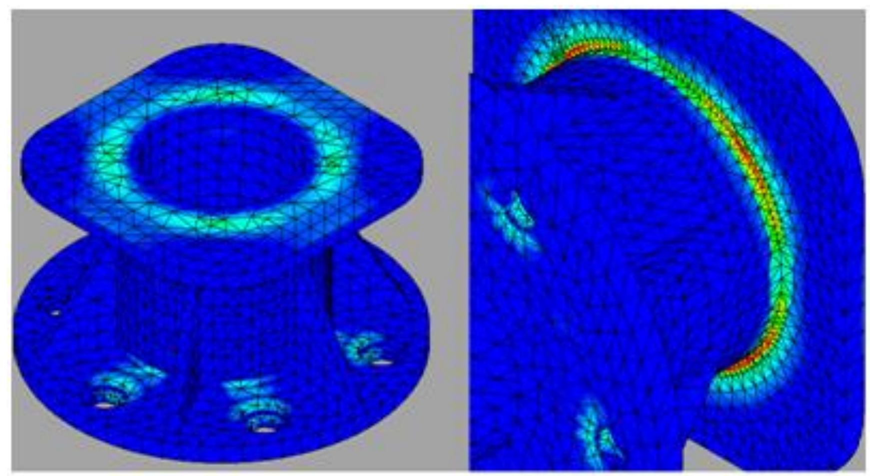
This stress can be considerably higher than the operational stress.

Applying a denser mesh around this simply leads to a much higher stress.

Dealing with singularities

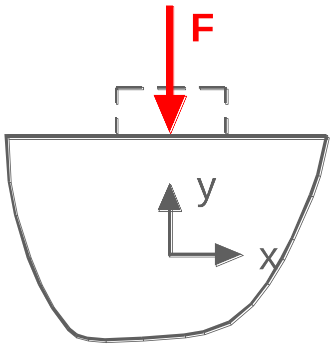
In FEA, stress singularities are a major concern when analyzing results, as they may considerably distort results. They are also a main cause for non-convergence of results in adaptive meshing.





Singularities of stresses and displacements indicate a deficiency of the *structural model* and not of FEM!

Dealing with singularities



Avoiding singularities by modeling

- Line or areal supports instead of point supports
- Line or areal loads instead of point loads
- Rounding of reentrant corners

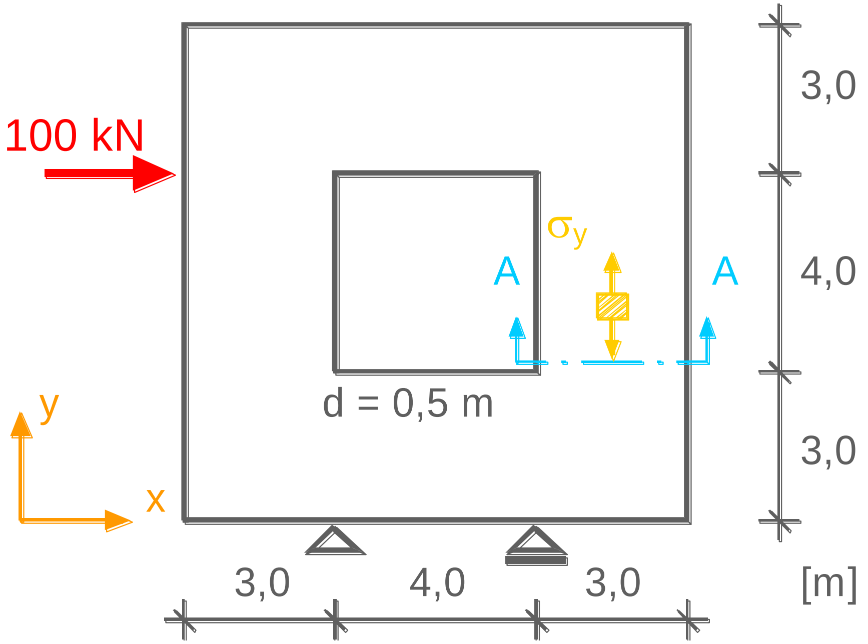
Accepting singularities / taking them into account at the result interpretation

- Stress results at the singularity points are physically meaningless
- Proper interpretation of the stresses at singular points: Integrated stress values instead of individual values
- Steel constructions: Limiting peak stresses e.g. in re-entrant corners by constructive measures

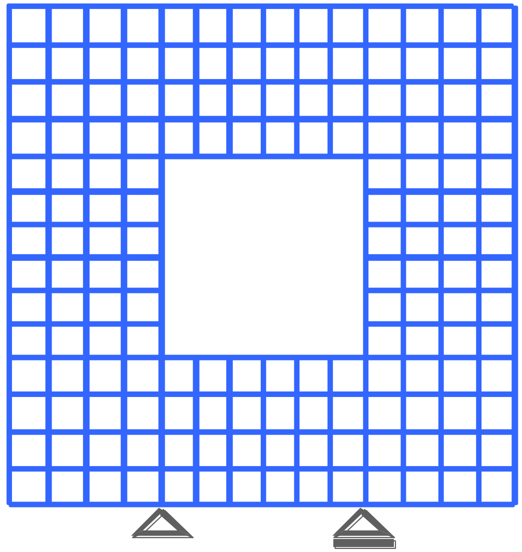
Applying special elements with singularities in the shape functions

Example: Plate in plane stress

Where are the points of singularities?



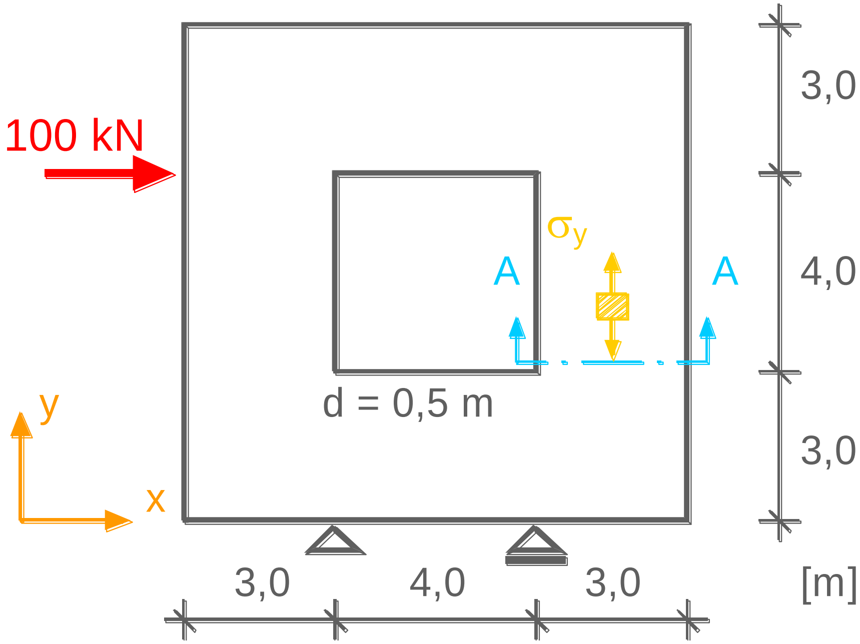
System



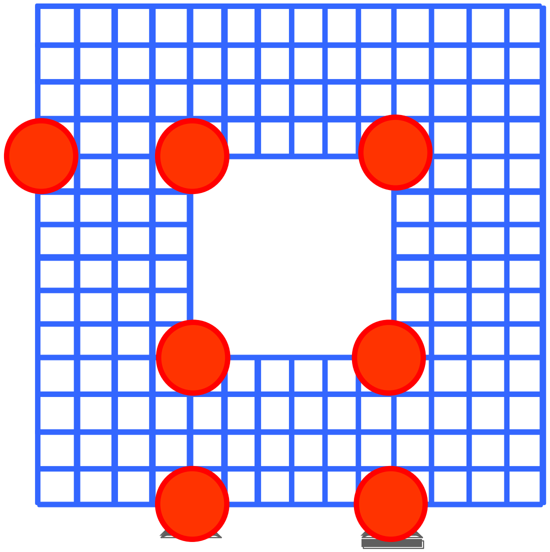
Finite element model (e = 75 [cm])

Example: Plate in plane stress

Where are the points of singularities?

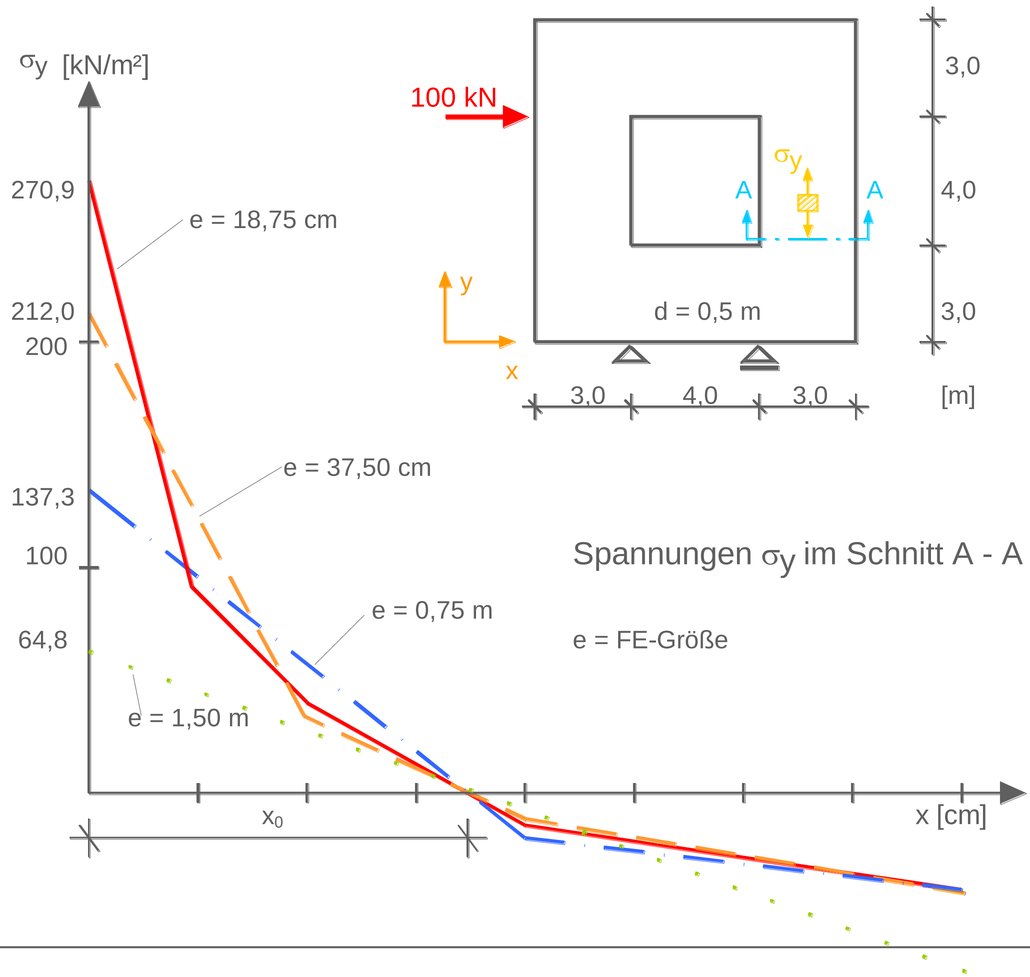


System



Finite element model (e = 75 [cm])

Stresses σ_y in section A-A for different finite element sizes



Results

- Stresses are converging in the middle of section A-A.
- In the corner point the stresses are not converging.
- In the neighbourhood of the corner stresses are „polluted" by the singularity.

➡ *Pollution effect*

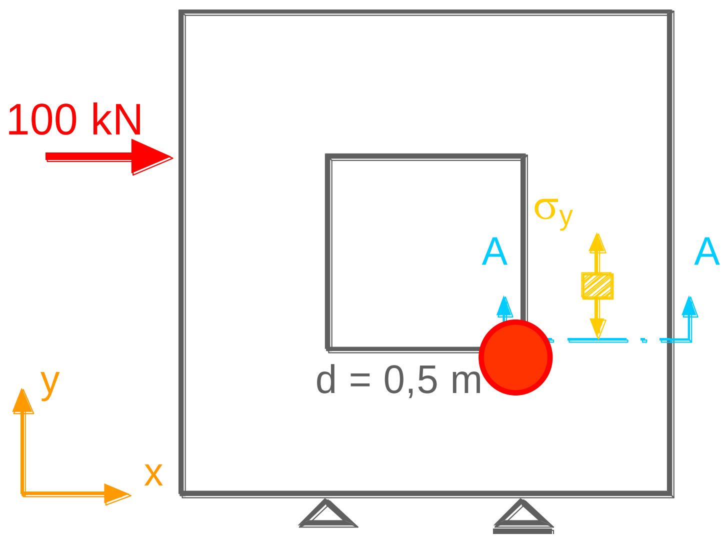
- Stress resultant of the tension stress converges and is nearly independent of element size.

Example: Plate in plane stress

Stresses in the corner point versus stress resultant of the tension stresses

MODEL	FE size e [cm]	Number of elements	Stress σ_y [kN/m²]	Distance x_0 [cm]	Stress resultant Z [kN]
HYBRID ELEMENTS	150.00	2	**64.8**	82	**13.3**
	75.00	4	**137.3**	66	**22.6**
	37.50	8	**212.0**	64	**24.9**
	18.75	16	**270.9**	64	**25.6**
ISOPARAMETR. ELEMENTS	150.00	2	27.4	52	3.6
	75.00	4	101.6	62	15.6
	37.50	8	180.6	62	21.2
	18.75	16	277.7	62	25.0

Stresses in the corner point versus stress resultant of the tension stresses

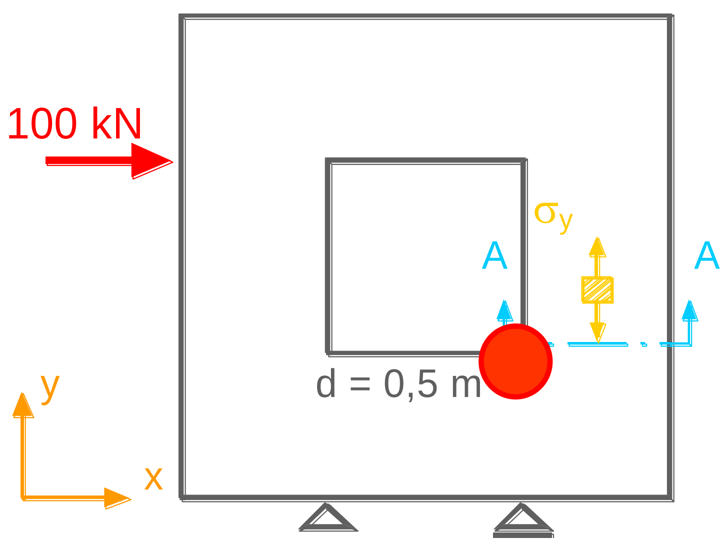


Results

- Stresses are converging in the middle of section A-A.
- In the corner point the stresses are not converging.
- In the neighbourhood of the corner stresses are „polluted“ by the singularity.

Stress resultant of the tension stresses converges and is nearly independent of element size.

Stresses in the corner point versus stress resultant of the tension stresses

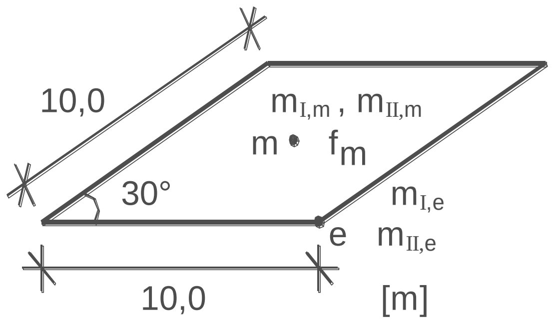


Results

- Stresses are converging in the middle of section A-A.
- In the corner point the stresses are not converging.
- In the neighbourhood of the corner stresses are „polluted“ by the singularity.

Stress resultant of the tension stresses converges and is nearly independent of element size.

Example: Skew plate in bending

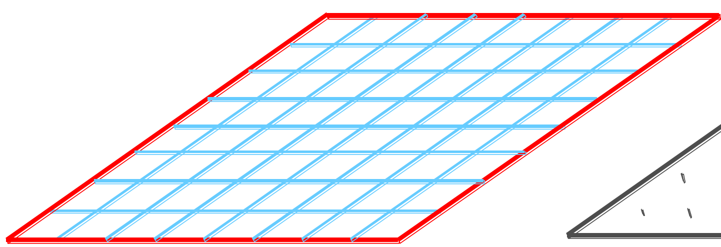


System

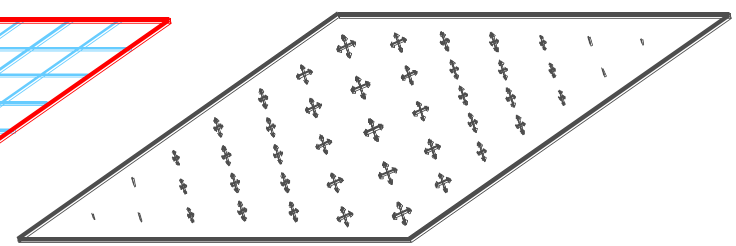
$q = 10 KN/m^2$

$E = 3 \cdot 10^7 \ kN/m^2$
$\mu = 0{,}3$
$d = 0{,}5 \ m$

Simply supported at all 4 sites



FE mesh 8 x 8



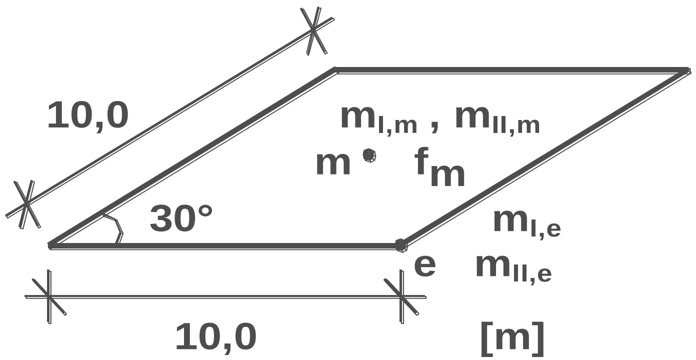
Principal moments

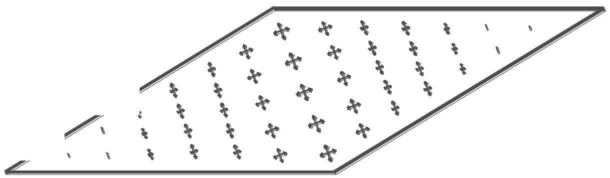
Example: Skew plate in bending

Displacements and principal moments of the plate

MODEL	f_m [mm]	$m_{I,m}$ [kNm/m]	$m_{II,m}$ [kNm/m]	$m_{I,e}$ [kNm/m]	$m_{II,e}$ [kNm/m]
Analytical	0.12	19.1	10.8	∞	-
Rigid supports					
2 x 2	**0.13**	**18.4**	12.0	**3.4**	2.3
4 x 4	**0.13**	**20.9**	13.3	**7.7**	0.3
8 x 8	**0.12**	**18.5**	10.8	**14.6**	1.1
16 x 16	**0.12**	**19.9**	11.5	**25.9**	1.6
32 x 32	-	-	-	**44.8**	1.8
Elastic supports					
4 x 4	-	**21.2**	12.9	**6.1**	0.4
8 x 8	-	**20.0**	11.8	**6.6**	0.1
16 x 16	-	**20.3**	12.0	**5.6**	-0.3

Example: Skew plate in bending

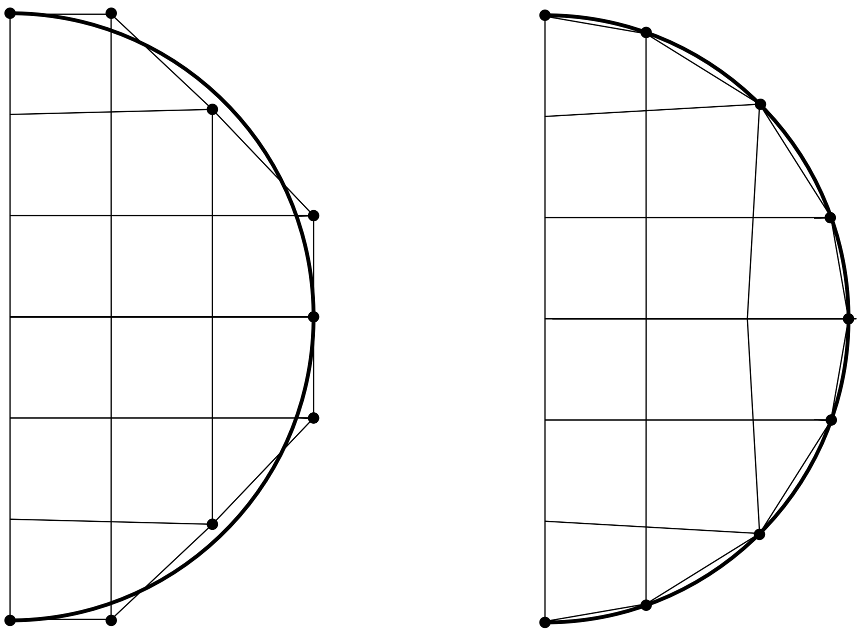




Result

- The structural model with a rigid support has a moment singularity in the obtuse corner.
- The structural model with an elastic rigid support has no moment singularity in the obtuse corner. The moments do converge.

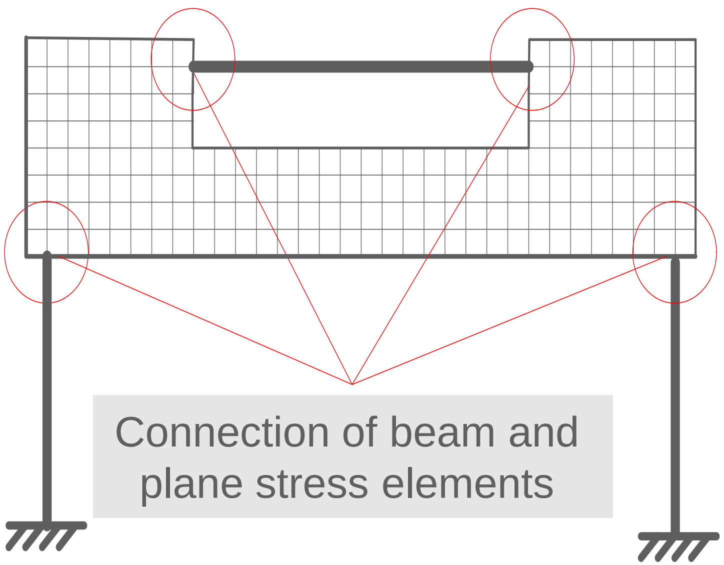
Geometric modeling of simply supported plates in bending

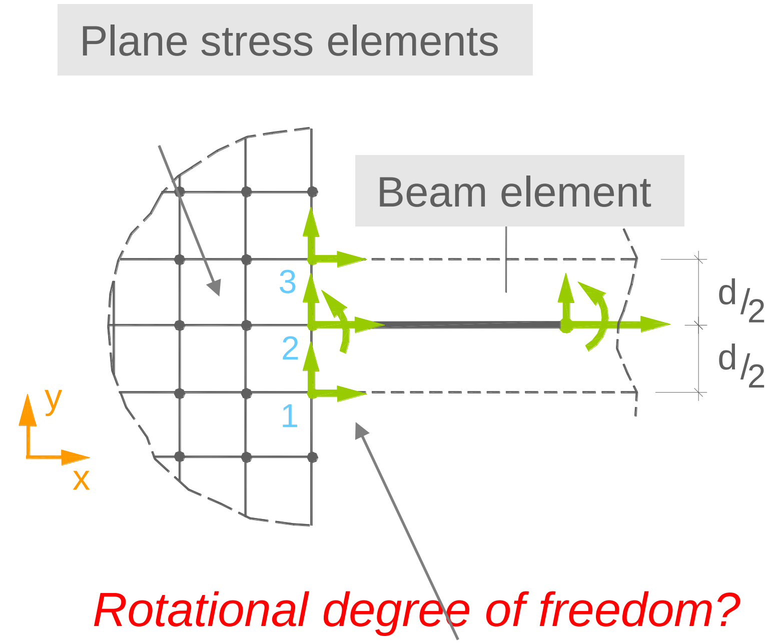


Node position approximated   Nodes exactly on a circle

A slight deviation from the circle geometry leads to a restraint effect at the edge!

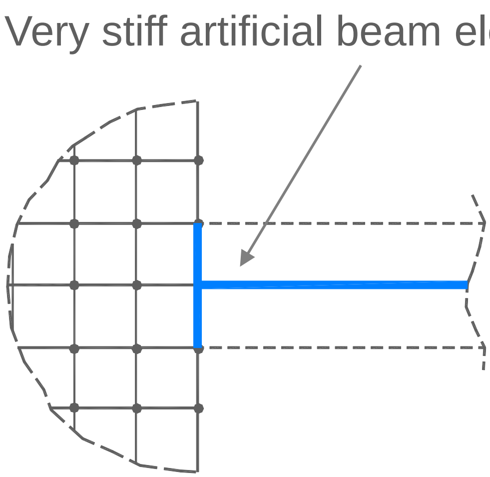
Connection of different types of structural elements



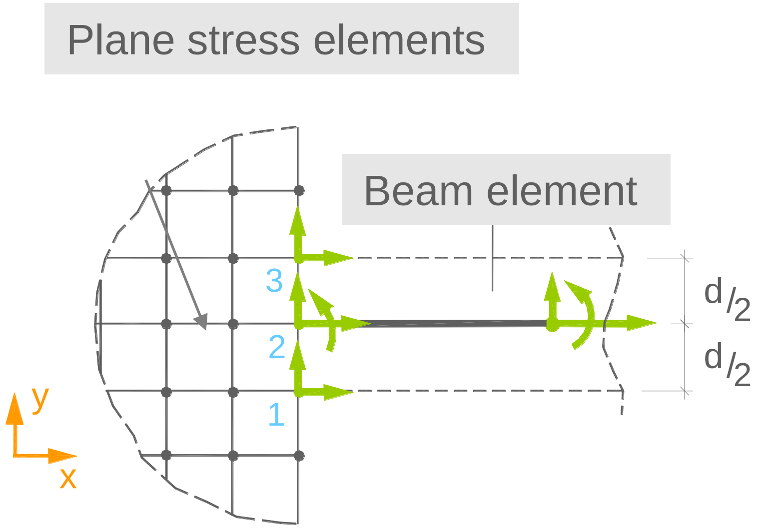


All degrees of freedom of both element types have to be connected.

Connection of different types of structural elements

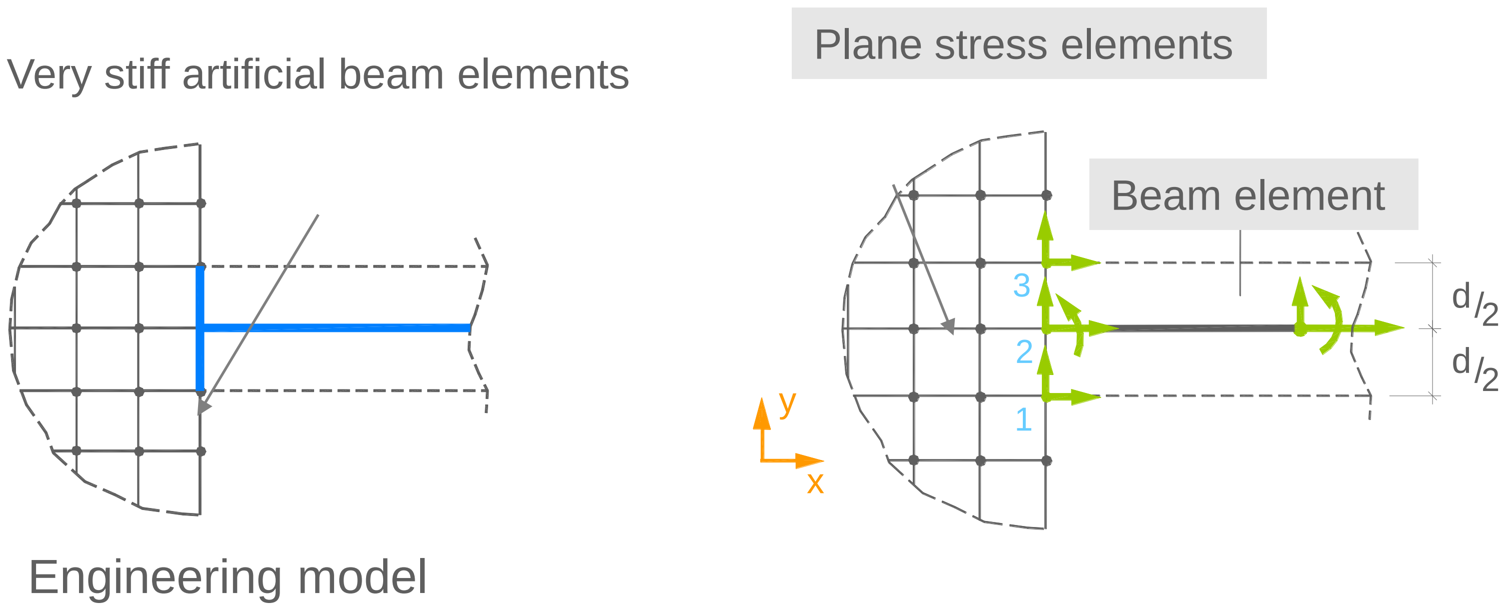


Engineering model



All degrees of freedom of both element types have to be connected.

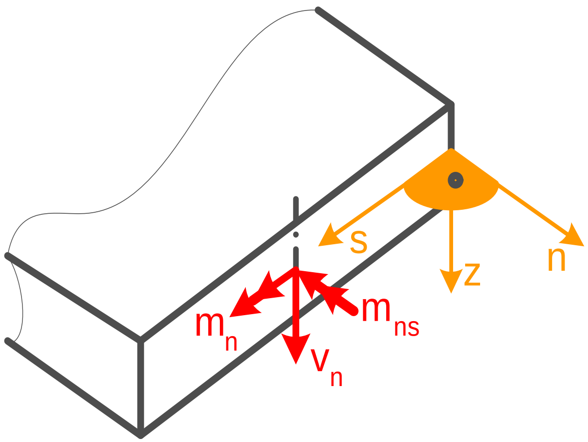
Connection of different types of structural elements



All degrees of freedom of both element types have to be connected.

Edge effect in plates in bending

Simply supported plate



Variation of the twisting moment at the edge.

Kirchhoff shear force: $\bar{v}_n = v_n + \frac{dm_{ns}}{ds}$

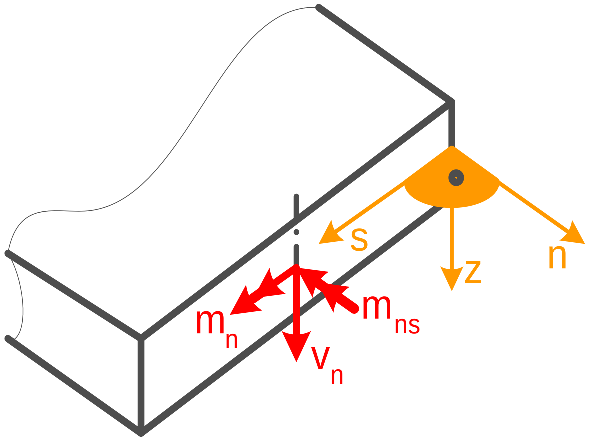
Support reaction

Shear force

The support reaction and the shear force at a *distance which is approximately equal to the plate thickness* are not the same for simply supported plates. They differ by the term dm_{ns}/ds .

Edge effect in plates in bending

Free edge



Kirchhoff shear force:

$$0 = v_n + \frac{dm_{ns}}{ds}$$

Variation of the twisting moment at the edge.

Shear force

The shear force at a *distance which is approximately equal to the plate thickness* equals the term dm_{ns} / ds, i.e. the variation of the twisting moment.

Edge effect in plates in bending

Finite element analysis

The section forces at the edges of a plate – *in a distance which is approximately equal to the plate thickness from the edge* – reflect the edge effect.

Thin plate with shear stiff elements (Kirchhoff plate theory)

Shear forces and twisting moments at a simply supported edge and a free edge reflect the edge effect, i.e. shear forces are not equal to the support reaction, twisting moments are not equal zero.

Thick plate with shear flexible elements (Reissner-Mindlin plate theory)

Shear forces and twisting moments at a simply supported edge and a free edge do not reflect the edge effect, if finite elements near the edge are very small, i.e. not larger than 1/5 of the plate thickness.

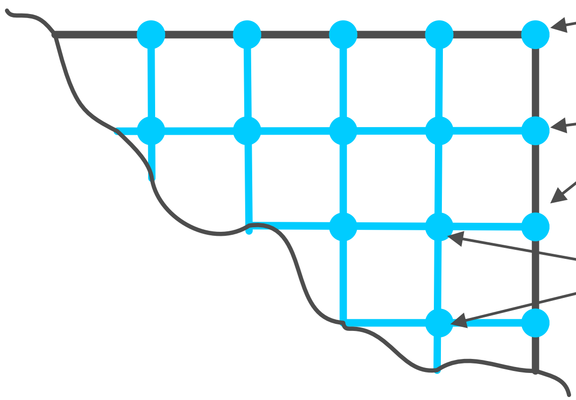
Interpretation of the results

Output points for sectional forces and stresses

- in the middle of the elements or in the integration points.
- in the nodal points by averaging of the element stresses of the adjacent elements.
- at an arbitrary point of the finite element mesh by interpolation of the values at the element centres or at the nodal points.

Interpretation of the results

Nodal stresses and section forces



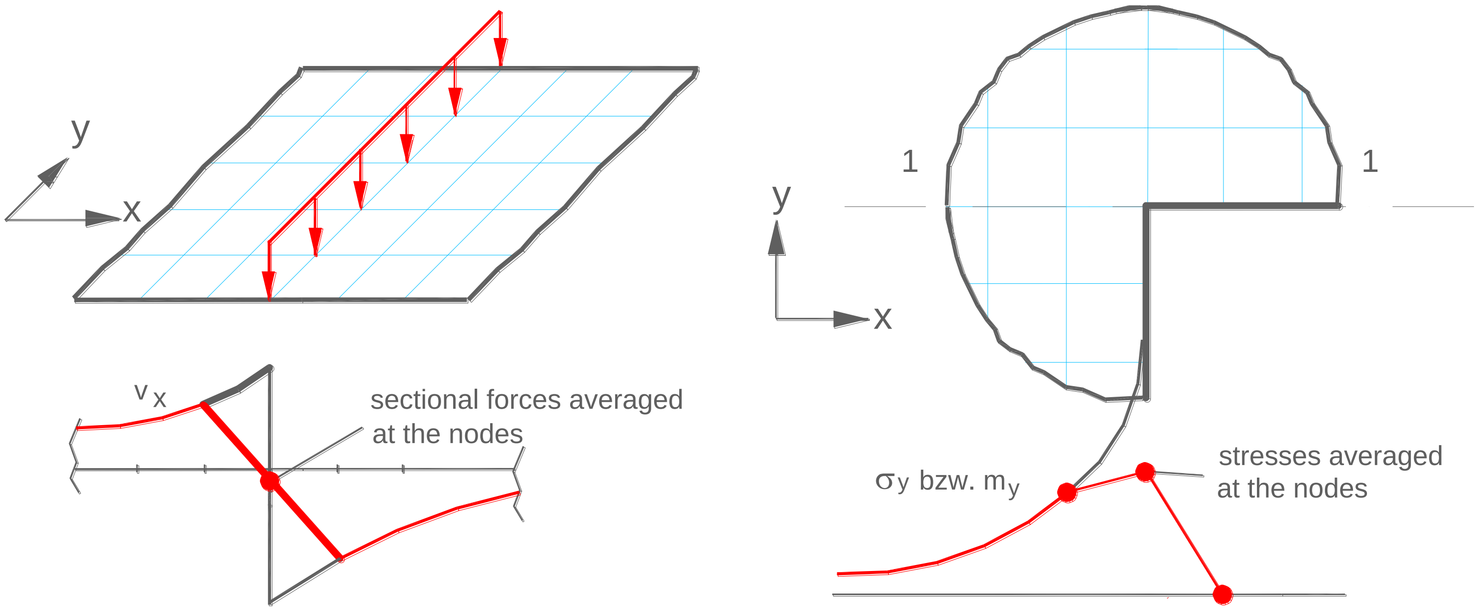
Nodal stresses are computed by averaging the element stresses at the node.

The more nodes are used the better the result.

4 elements: good / 2 elements: acceptable / 1 element: no averaging

Interpretation of the results

Output points for sectional forces and stresses



Jumps of sectional forces and stresses in plate structures

Error estimation and adaptive meshing

Discretization error: Error caused by the approximation of the exact displacements by shape functions.

$$\underline{e} = \underline{u}^{(FE)} - \underline{u}^{(exakt)}$$

Global error: Discretization error of the total system – energy norm.

Locale error: Discretization error of sectional forces and displacements at individual points.

Error estimation: Upper limit of the error (proved mathematically).

Methods for the estimation of the global error are available.

The local error cannot be estimated efficiently so far.

Example

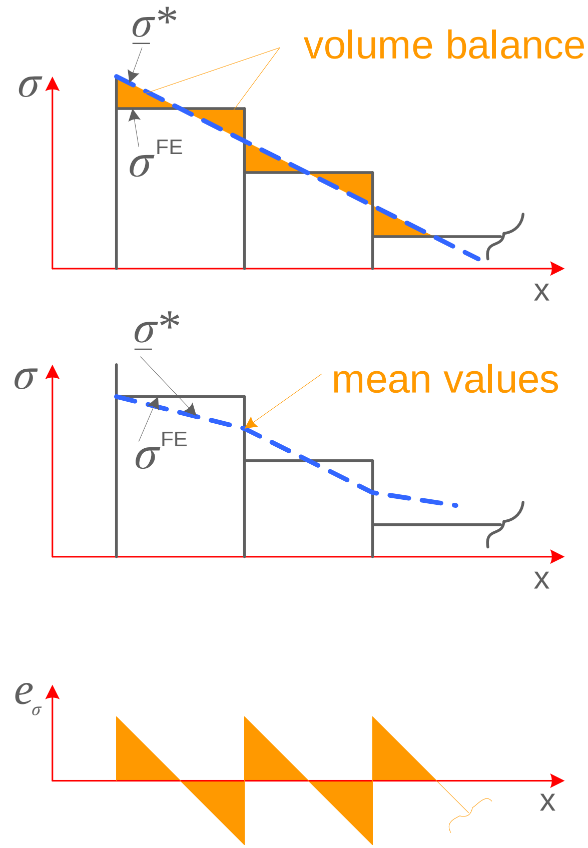
Error estimation and adaptive meshing

Error estimation according to Zienkiewicz/Zhu

Step 1: Smoothing of stress distribution

Simplified smoothing of stresses:

- Averaging of the element stresses at the nodes
- Linear interpolation between the nodes



Step 2: Error estimation

$e_\sigma = \sigma^* - \sigma^{FE}$

σ^* = improved stresses acc. to step 1

σ^{FE} = stresses in the finite element

Step 3: Mean error in the element

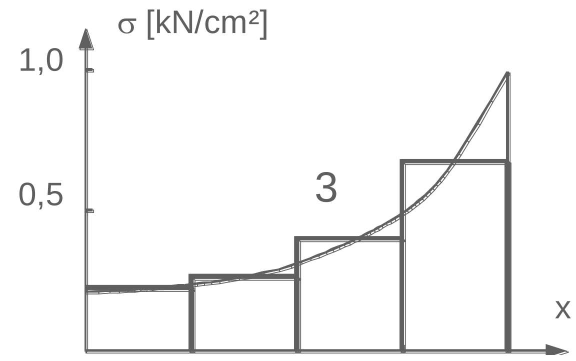
$$\|e_\sigma\| = \alpha \cdot \sqrt{\frac{\int_A \left(\sigma^* - \sigma^{FE}\right)^2 dA}{A}}$$

A= element area $\alpha = 1.1$ for rectangular plane stress element

Error estimation according to Zienkiewicz/Zhu

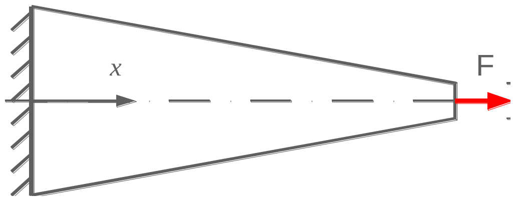
Example: Error in element 3

$$\|e_\sigma\| = 1.1 \cdot \sqrt{\frac{\int_A \left(\sigma^* - \sigma^{FE}\right)^2 dA}{A}}$$



Stress σ*(x) and beam height h(x)
at the integration points:

x = 276.4 cm h = 27.89 cm σ* = 0.383 kN/cm²
x = 348.6 cm h = 22.11 cm σ* = 0.494 kN/cm²



Error:

Computation | Gauss

$$\|e_\sigma\| = 1.1 \cdot \sqrt{\frac{[(0.383-0.400)^2 \cdot 27.89 + (0.494-0.400)^2 \cdot 22.11] \cdot 125 \cdot 0.5}{3125}} = 0.07$$

Error estimation and adaptive meshing

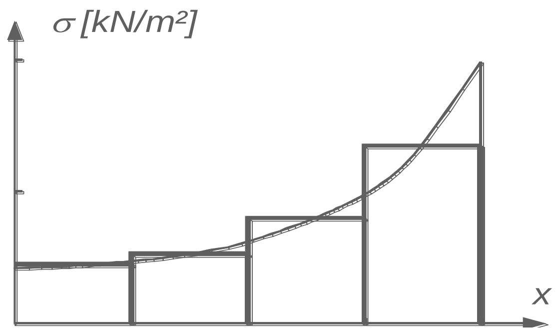
Example: Truss element with linearly varying cross section area

Shape functions	Number of elements	x [cm]				
		0	125	250	375	500
linear	1	0.333	-	0.333	-	0.333
	2	0.250	0.250	0.250/0.500	0.500	0.500
	4	0.222	0.222/0.286	0.286/0.400	0.400/0.667	0.667
quadratic	1	0.130	-	0.391	-	0.652
	2	0.191	0.255	0.319/0.273	0.545	0.818
	4	0.198	0.248/0.247	0.329/0.324	0.486/0.462	0.923
exact	-	0.200	0.250	0.333	0.500	1.000

Element stresses in the example [kN/cm^2]

Error estimation and adaptive meshing

Error estimation according to Zienkiewicz/Zhu



Result

The local accuracy of a finite element analysis is given by the error estimator. A mathematically exact error bound, however, cannot be expected.

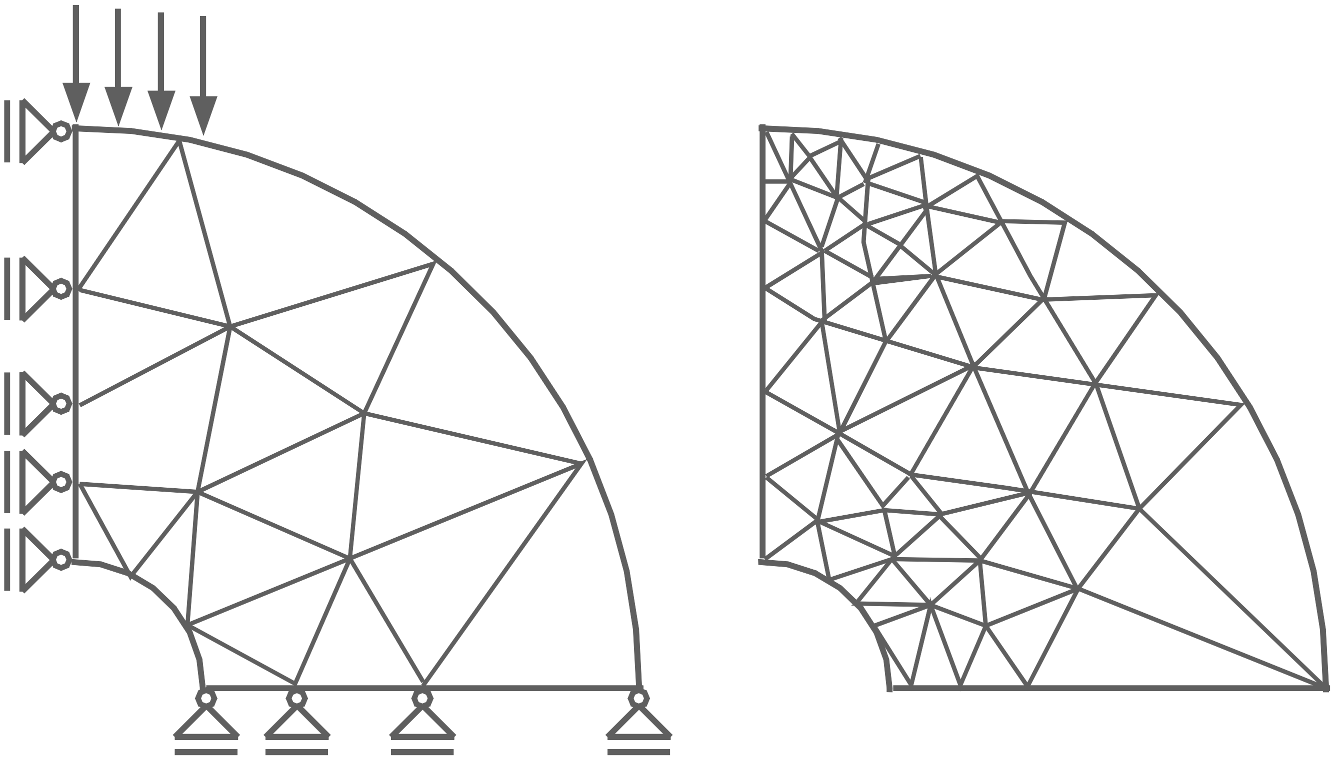
STRESS OUTPUT POINT	x [cm]				
	0	125	250	375	500
Element	0.222	0.286	0.400	0.667	
Node	0.222	0.254	0.343	0.534	0.667
$\lVert e_\sigma \rVert$	0.014	0.025	0.070	0.114	

Node and element stresses

Error of FEM solution

Error estimation and adaptive meshing

Adaptive meshing



In an adaptive meshing the finite element net is refined where the local error is large.

Example of an adaptive meshing

Controlling of finite element computations

Controlling strategy for FE results

- **First check: Overview of the results**
  - Checking the plausibility of the displacements (graphical)
  - Qualitative assessment of the distribution of sectional forces and stresses
  - Check of the sum of loads of all load cases

- **Final check: Checking the details**
  - Checking all input data for structural analysis in detail
  - Approximate calculation of significant values of sectional forces and displacements (by hand).

Controlling of finite element computations

A Finite Element Analysis should be part of the quality assurance process in an Engineering project.

Three principal aims of Quality Assurance Process :

- a clear definition of what is to be achieved
- a description of the activities and functions that need to be performed
- the control and monitoring of the performance of those activities and functions.

End

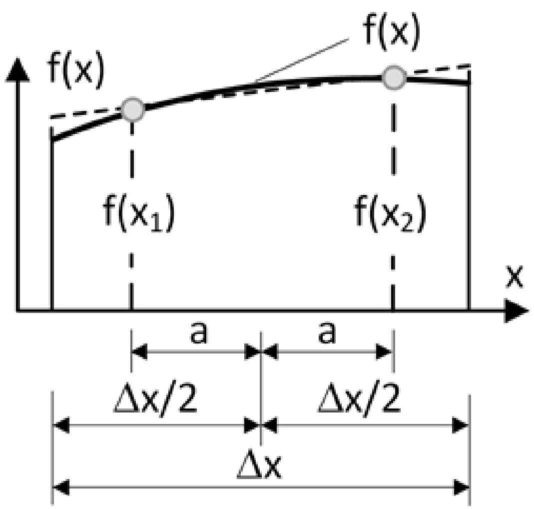
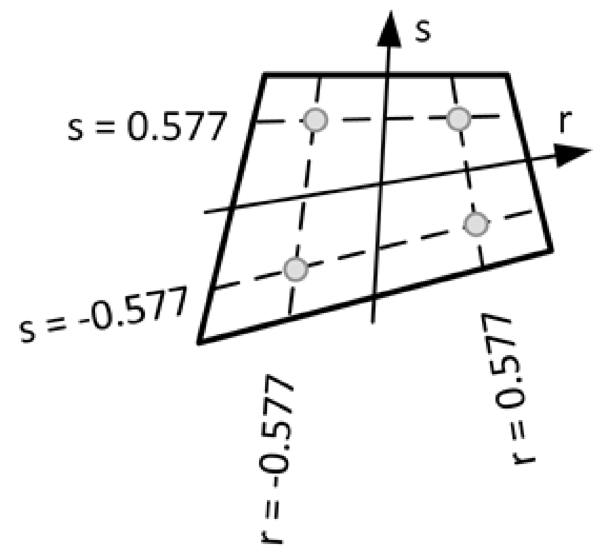
Introduction

Truss and beam structures

Plate and shell structures

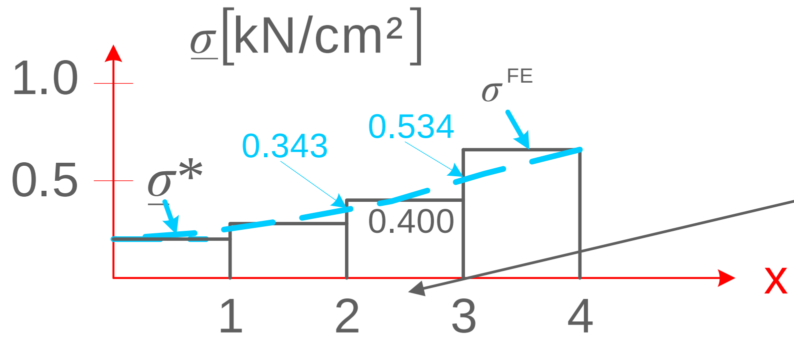
4 Modeling and quality assurance

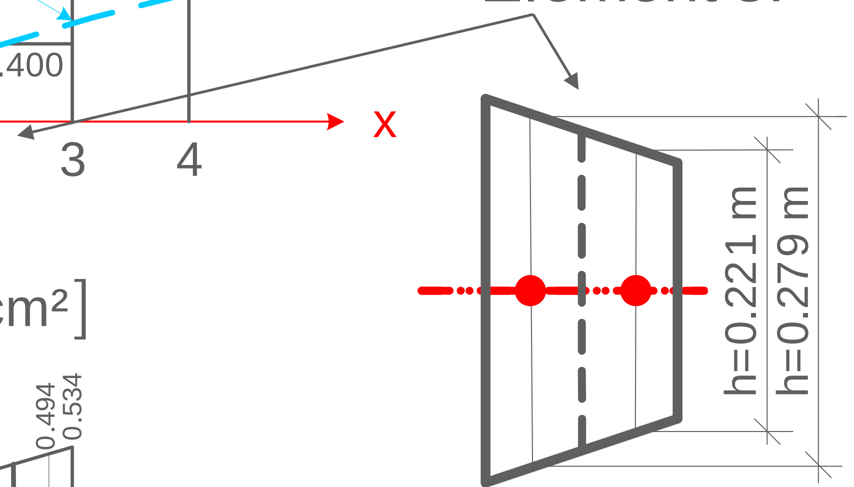
Gauss integration

Integration order n	Formula $\int_{x_a}^{x_a+\Delta x} f(x)\ \mathrm{d}x = \sum_i f(x_i)\cdot \alpha_i\ \frac{\Delta x}{2}$	Location of integration points $r = r_j$ and $s = s_k$ in plane finite elements
2-point integration  $n=2,\ a = \xi_1 \cdot \frac{\Delta x}{2}$	$\int_{x_a}^{x_a+\Delta x} f(x)\,\mathrm{d}x =$ $\left(\alpha_1 f(x_1) + \alpha_2 f(x_2)\right)\cdot\frac{\Delta x}{2}$ $\alpha_1 = 1 \qquad \alpha_2 = 1$ $\xi_1 = 1/\sqrt{3} \approx 0.577$ 3rd degree polynomial is integrated exactly	

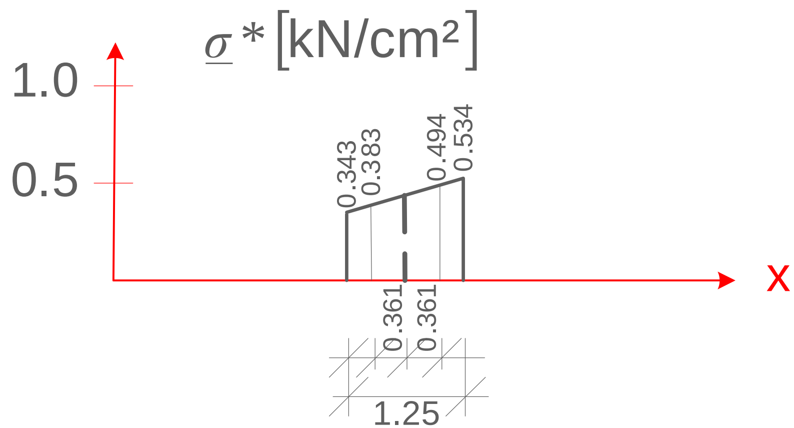
Error estimation and adaptive meshing

Averaged nodal stresses





Nodal stresses at the Gauss points in element 3



Element area: $A = \int h(x)\,dx = 0.5 \cdot (0.221 + 0.279) \cdot 1.25 = 0.3125\ [m^2]$

Error estimation and adaptive meshing

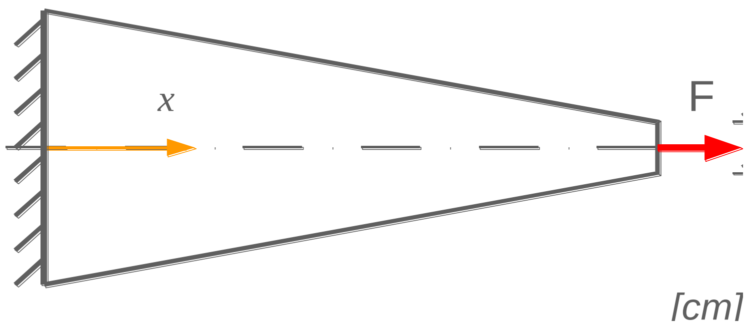
Example: Truss element with linearly varying cross section area

Shape functions	Number of elements	x [cm]				
		0	**125**	**250**	**375**	**500**
linear	1	0.133	-	0	-	0.667
	2	0.050	0	0.083/0.167	0	0.500
	4	0.022	0.028/0.036	0.047/0.067	0.100/0.167	0.333
quadratic	1	0.070	-	0.058	-	0.348
	2	0.009	0.005	0.014/0.060	0.045	0.182
	4	0.002	0.002/0.003	0.004/0.009	0.014/0.038	0.077

Error in the element stresses in the example [kN/cm^2]
Maximum stress: 1 [kN/cm^2]

Error estimation and adaptive meshing

Truss element with variable cross section area



Parameters

A_1 = 500 cm^2
A_2 = 100 cm^2
E = 1000 kN/cm^2
F = 100 kN