Finite Elements in Structural Analysis

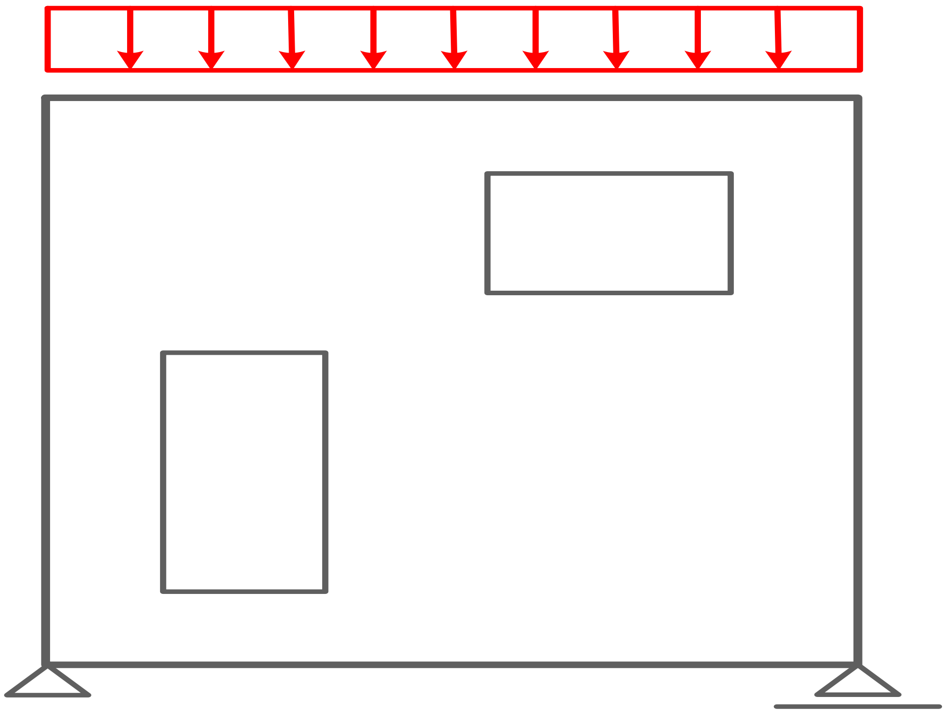
Theory of Elasticity

1 Plates in plane stress

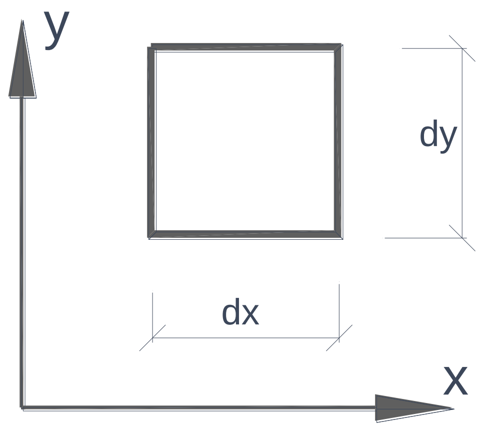
2 Plates in bending

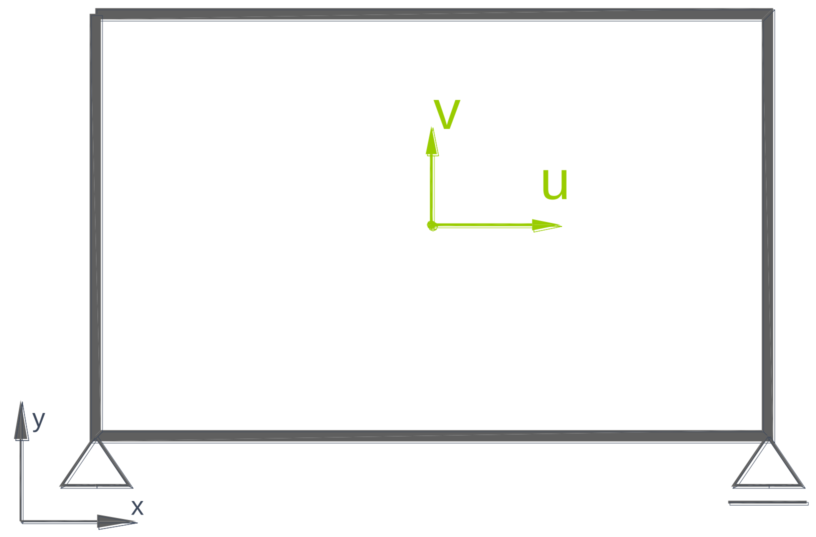
Definition

A plate in plane stress is a plane structure which is loaded or exposed to stresses in its plane.

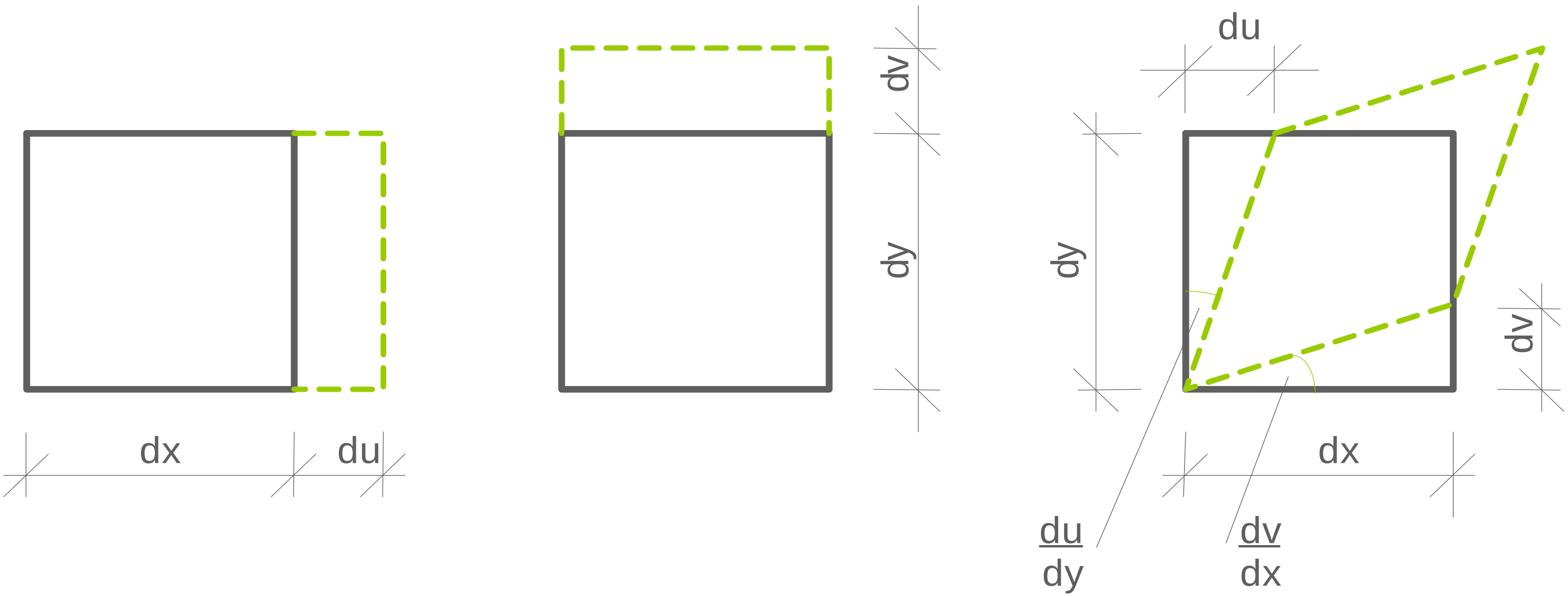


Displacements: u, v



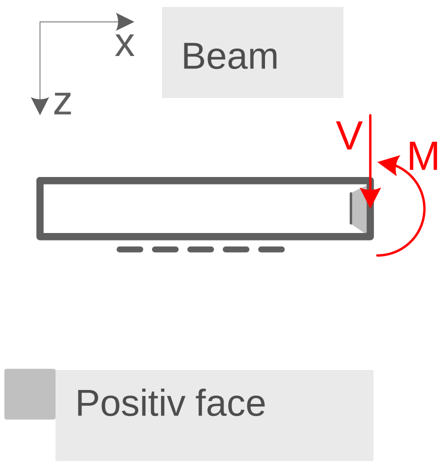


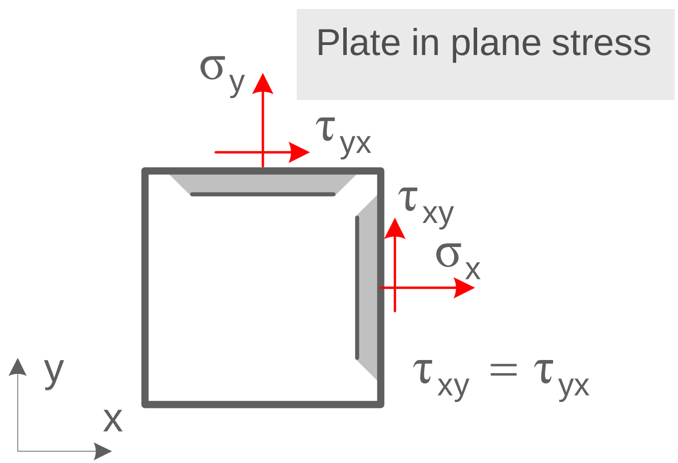
Strains and shear angles: $\varepsilon_x, \varepsilon_y, \gamma_{xy}$



$$\varepsilon_x = \frac{du}{dx} \qquad \varepsilon_y = \frac{dv}{dy} \qquad \gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx}$$

Sign convention of Stresses and section forces





Sign convention of section forces in Beams:

Section forces in beams and shear forces in plates are positive when they act in the positive coordinate direction at the positive face of an element.

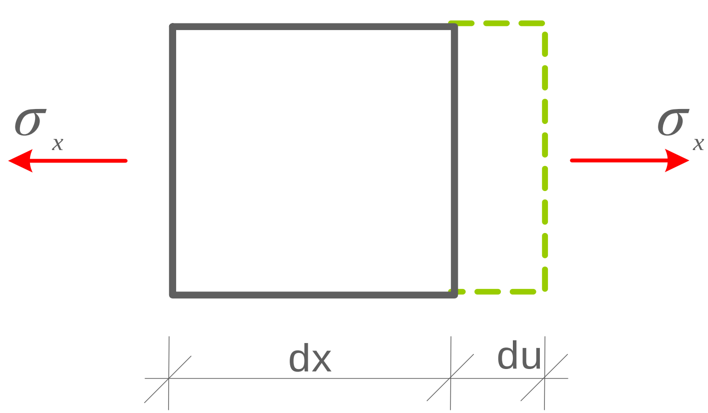
Sign convention of stresses:

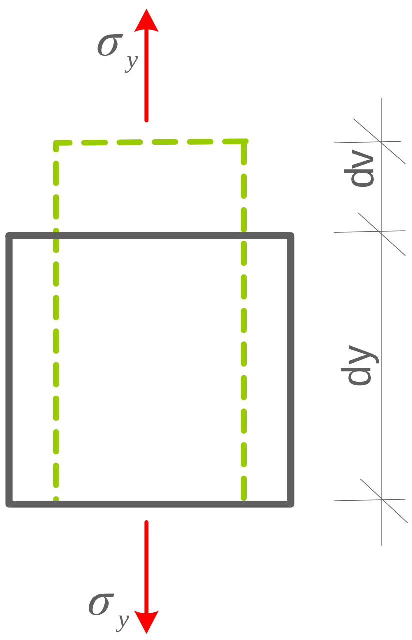
Stresses are positive when they act in the positive coordinate direction at the positive face of an element. At a positive face the outward normal vector is in direction of the positive coordinate.

Material law

Hooke's law:

$$\underline{\sigma} = \underline{D} \cdot \underline{\varepsilon}$$





Strain in x - direction:

$$\varepsilon_x = \frac{\sigma_x}{E} - \mu\,\varepsilon_y = \frac{\sigma_x}{E} - \mu \cdot \frac{\sigma_y}{E}$$

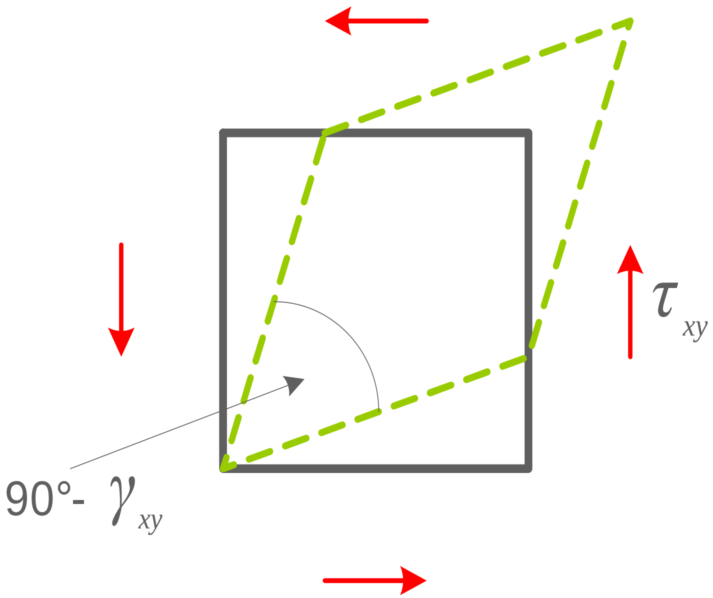
Linear strain ($\frac{\sigma_x}{E}$)

Transverse strain ($\mu\,\varepsilon_y$)

Strain in y - direction:

$$\varepsilon_y = -\mu \cdot \frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

Material law



Shear angle:

$$\tau_{xy} = G \cdot \gamma_{xy} \quad \rightarrow \quad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$G = \frac{E}{2(1+\mu)} \quad \text{or} \quad = \frac{2(1+\mu)}{E} \cdot \tau_{xy}$$

Material law

Strains due to Stresses:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \cdot \begin{bmatrix} 1 & -\mu & 0 \\ -\mu & 1 & 0 \\ 0 & 0 & 2(1+\mu) \end{bmatrix} \cdot \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Hooke's law:

(after matrix inversion or solving of equations for σ_x , σ_y , τ_{xy})

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\mu^2} \cdot \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

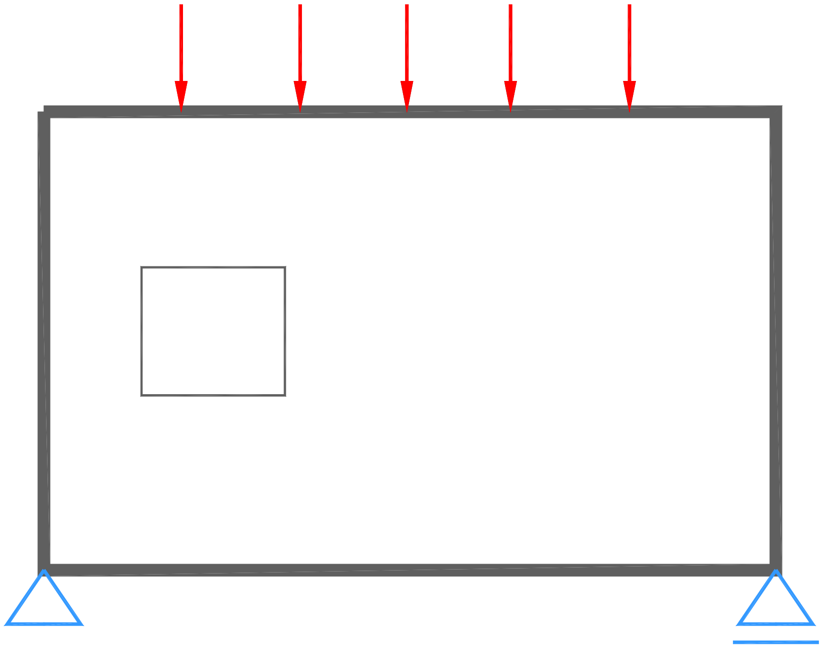
$$\underline{\sigma} = \underline{D} \cdot \underline{\varepsilon}$$

Two-dimensional material law

Plane stress

Stresses normal to the plane of the plate are zero.

Example:
Reinforced concrete deep beam



Two-dimensional material law

Plane stress

$(\boldsymbol{\sigma}_z = 0)$

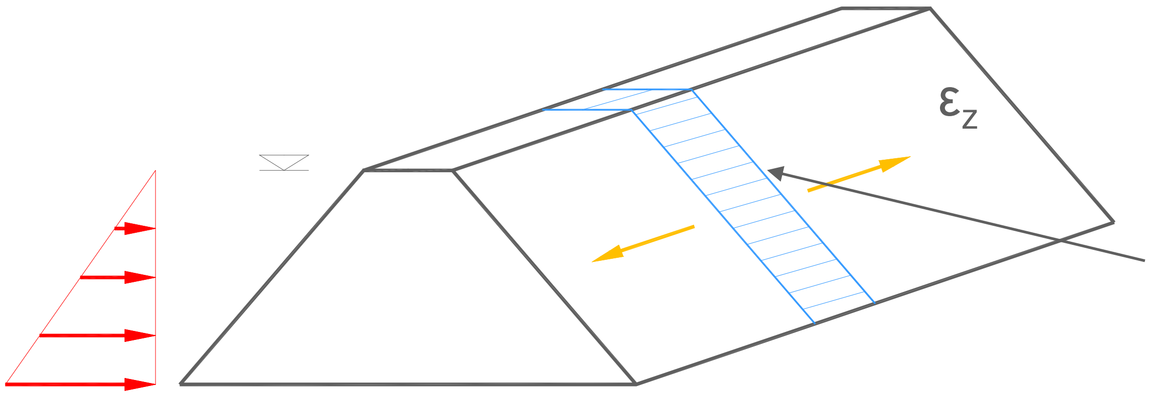
$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\mu^2} \cdot \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Two-dimensional material law

Plane strain

Strain normal to the plane is zero

Example: Dam



$\varepsilon_z = 0$

Two dimensional slice of a dam

Two-dimensional material law

Plane strain

$(\varepsilon_z = 0)$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \cdot \begin{bmatrix} 1 & \frac{\mu}{(1-\mu)} & 0 \\ \frac{\mu}{(1-\mu)} & 1 & 0 \\ 0 & 0 & \frac{1-2\mu}{2(1-\mu)} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Two-dimensional material law

Plane strain

Hooke‘s law for orthotropical materials

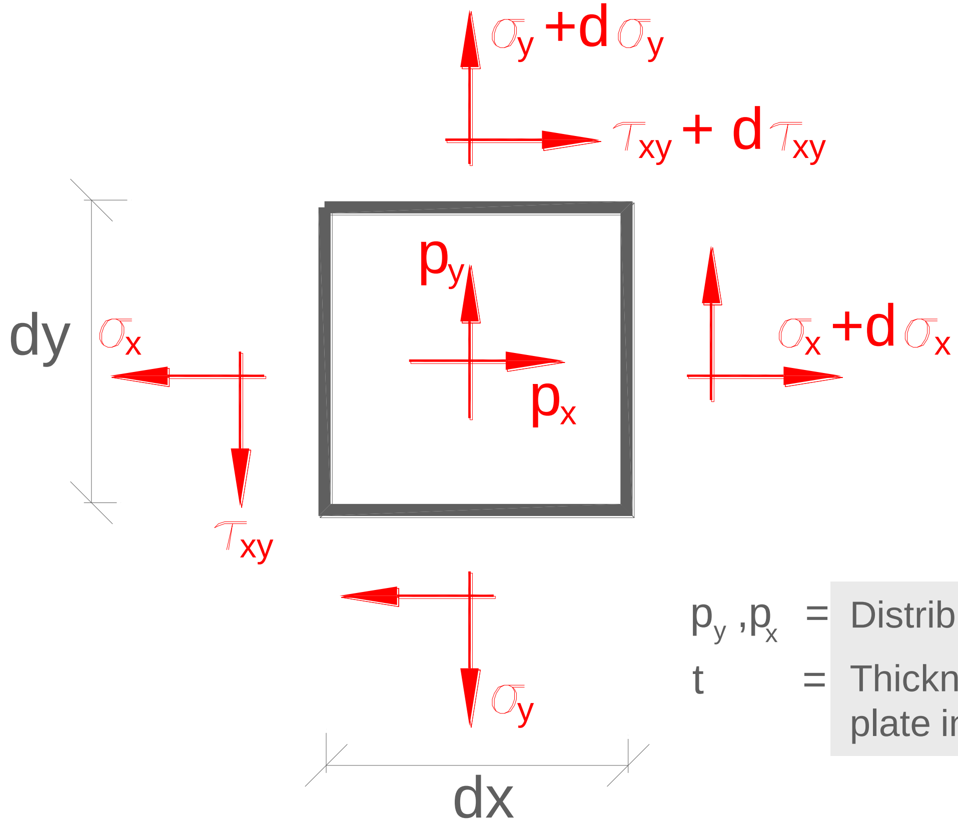
Definition
Orthotropic materials possess different elastic properties (E, G) in two perpendicular directions.

Hooke’s law
- plane stress
- plane strain

Examples
- Wood
- Orthotropic soils

Equilibrium conditions

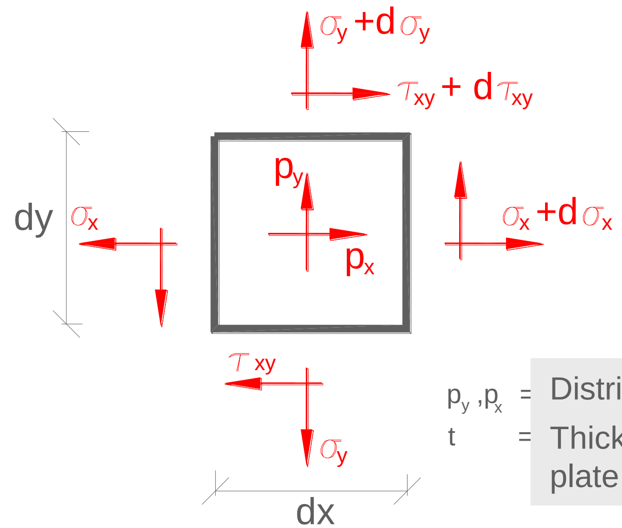
Infinitesimal element



p_y ,p_x = Distributed load

t = Thickness of the plate in plane stress

Equilibrium conditions



p_y, p_x = Distributed load
t = Thickness of the plate in plane stress

$$\sum X = 0$$

$$-\sigma_x \cdot t \cdot dy + (\sigma_x + d\sigma_x) \cdot t \cdot dy - \tau_{xy} \cdot t \cdot dx + (\tau_{xy} + d\tau_{xy}) \cdot t \cdot dx + p_x \cdot dx\, dy = 0 \qquad (1)$$

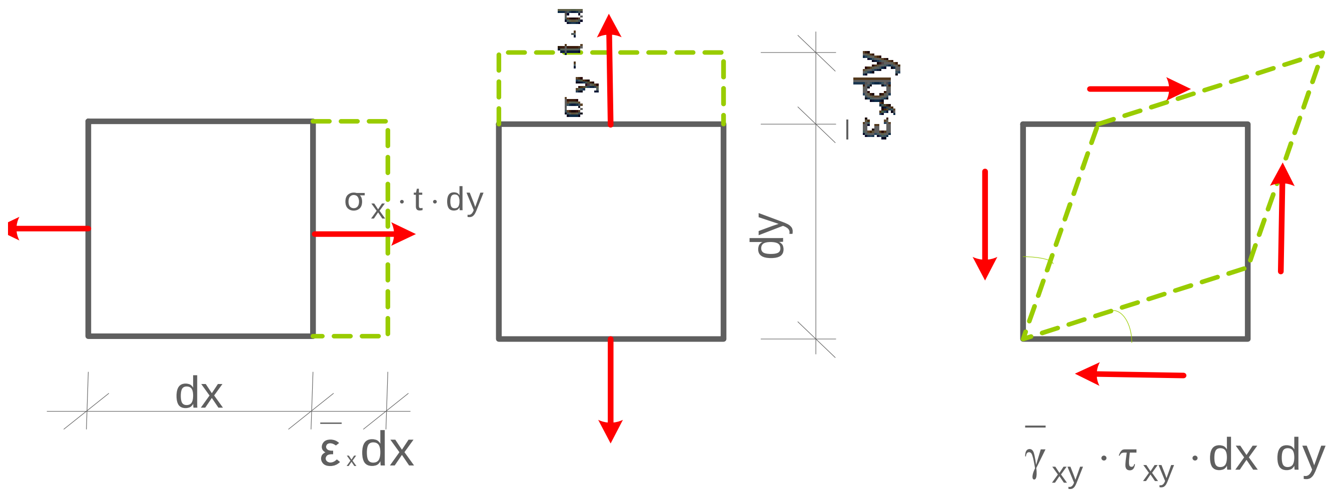
$$\sum Y = 0$$

$$-\sigma_y \cdot t \cdot dx + (\sigma_y + d\sigma_y) \cdot t \cdot dx - \tau_{xy} \cdot t \cdot dy + (\tau_{xy} + d\tau_{xy}) \cdot t \cdot dy + p_y \cdot dx\, dy = 0 \qquad (2)$$

of (1) $$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{p_x}{t} = 0$$

of (2) $$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{p_y}{t} = 0$$

Principle of virtual displacements



Example:

Strain in x-direction

Force : $\sigma_x \cdot t \cdot dy$

Virtual displacement : $\bar{\varepsilon}_x \cdot dx$

Internal work : $t \cdot \sigma_x \cdot \bar{\varepsilon}_x \cdot dx \; dy$

$$\bar{W}_i = \bar{W}_a$$

$$\bar{W}_i = t \cdot \int \left(\sigma_x \, \bar{\varepsilon}_x + \sigma_y \, \bar{\varepsilon}_y + \tau_{xy} \, \bar{\gamma}_{xy} \right) \cdot dx \, dy$$

Principle of virtual displacements

$$\bar{W}_i = t \cdot \int \left(\sigma_x \, \bar{\varepsilon}_x + \sigma_y \, \bar{\varepsilon}_y + \tau_{xy} \, \bar{\gamma}_{xy} \right) \cdot dx \, dy$$

$$\bar{W}_i = t \cdot \int \left[\bar{\varepsilon}_x \quad \bar{\varepsilon}_y \quad \bar{\gamma}_{xy} \right] \cdot \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dx \, dy$$

$\sigma_x , \sigma_y , \tau_{xy}$ = real stress

$\bar{\varepsilon}_x , \bar{\varepsilon}_y , \bar{\gamma}_{xy}$ = virtual strain

t = thickness of the plate

$$\bar{W}_i = t \cdot \int \bar{\varepsilon}^T \sigma \, dx \, dy = t \int \bar{\varepsilon}^T D \, \varepsilon \, dx \, dy$$

Notation: $\bar{W}_i$ in one dimensional case

$$W_i = \int \underbrace{A \cdot \sigma_x}_{N} \cdot \underbrace{\bar{\varepsilon}_x}_{\frac{\bar{N}}{EA}} \cdot dx = \int \frac{N\bar{N}}{EA} dx$$

with $\bar{\varepsilon}_x = \frac{\bar{\sigma}_x}{E} = \frac{\bar{N}}{A} \cdot \frac{1}{E} = \frac{\bar{N}}{EA}$

End

1 Plates in plane stress

Plates in bending