Finite Elements in Structural Analysis

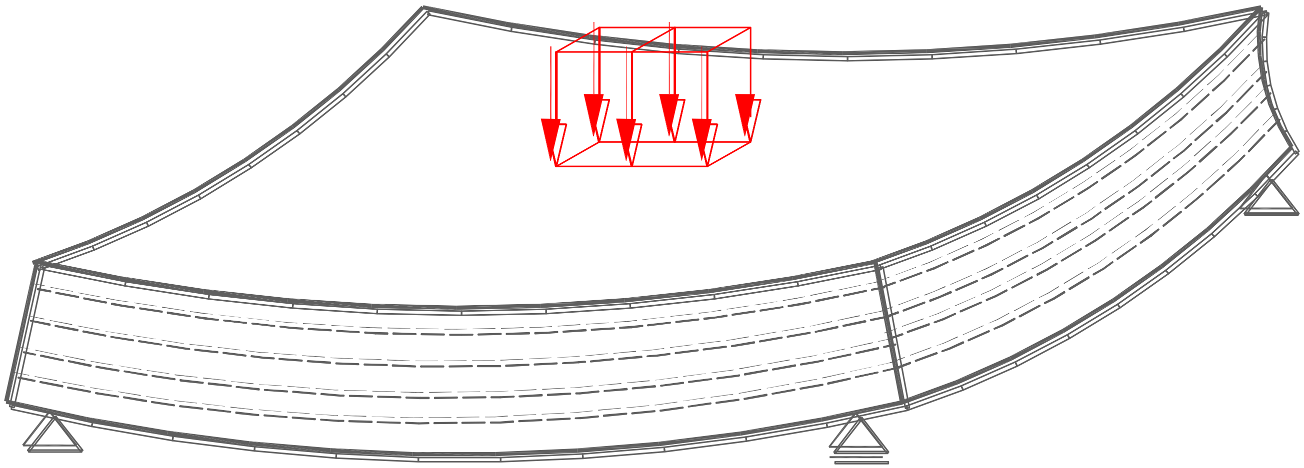
Theory of Elasticity

1 Plates in plane stress

2 Plates in bending

Definition

A plate in bending is a plane structure which is loaded perpendicularly to its plane.



Load-bearing behaviour

Beam

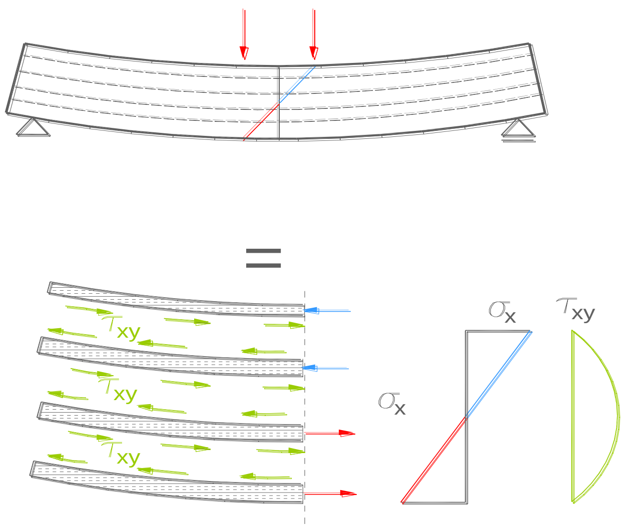
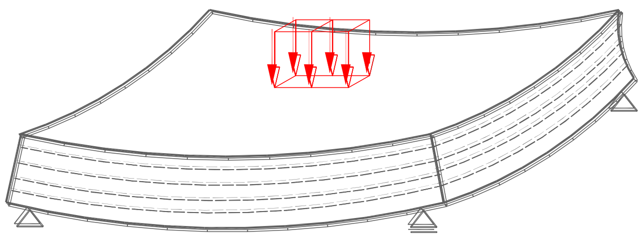


Plate in bending



Plates in bending:
=„glued package of plates in plane stress“.

Plates in plane stress:
Normal stresses σ_x, σ_y, and
shear stresses τ_{xy} - In-plane shear

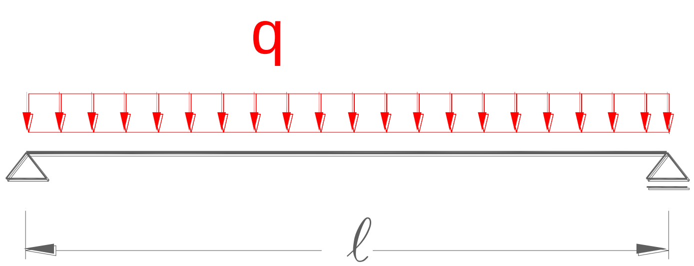
„Glued package“:
Shear stresses τ_{xz}, τ_{yz} – Bending shear

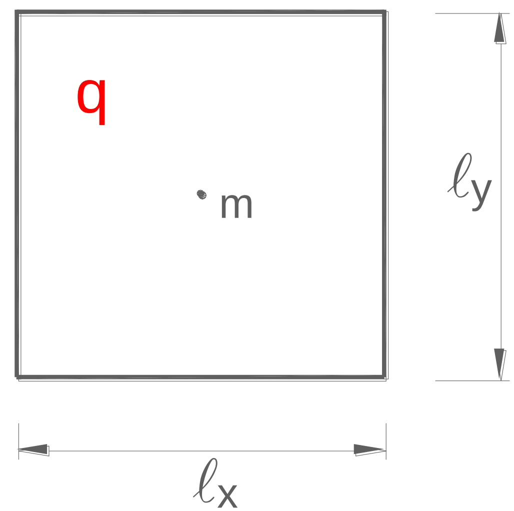
Beam = „glued package of fibres“

Fibres: Normal stresses σ_x

„Glued package“: Shear stresses τ

Moments





$$\frac{\ell_y}{\ell_x} = 1$$

Simply supported

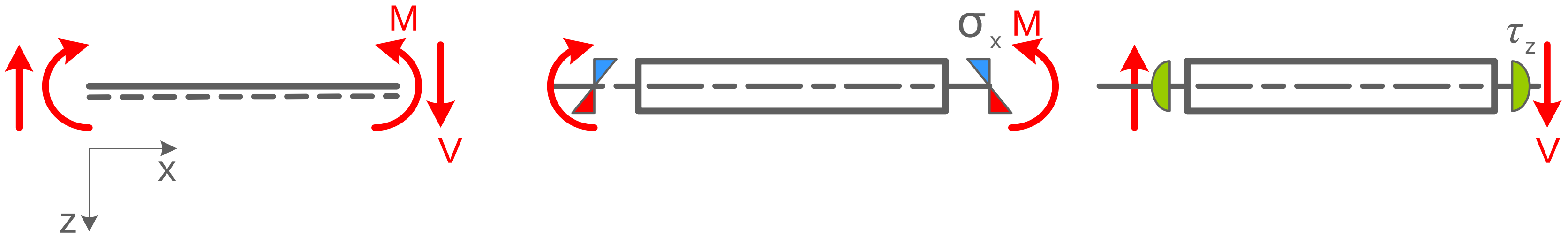
$$M_m = \frac{q\ell^2}{8}$$

$$m_{xm} = m_{ym} = \frac{q\ell^2}{27}$$

Section forces and stresses in beams and plates in bending

Beams

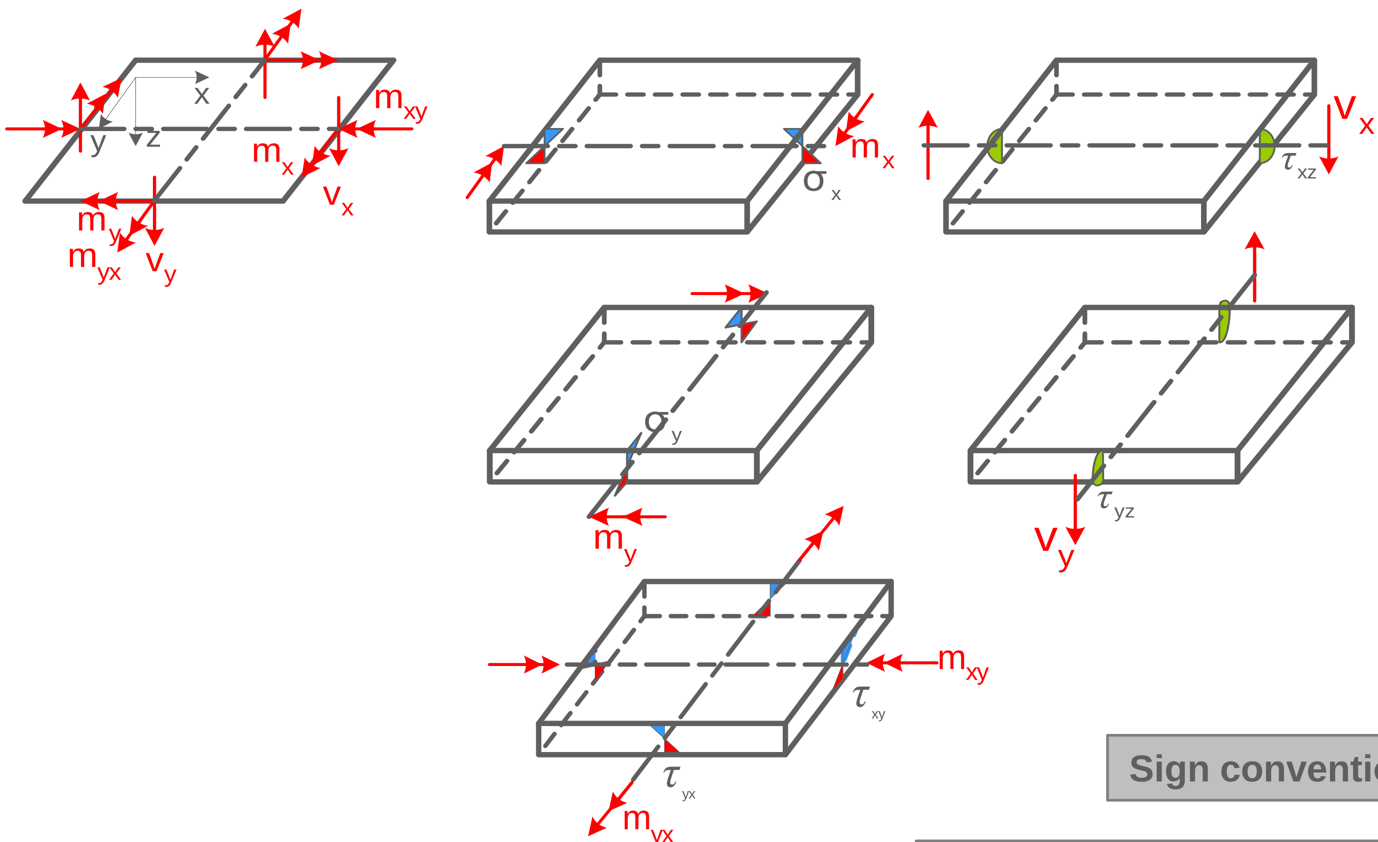
System Moments Shear forces



Section forces and stresses in beams and plates in bending

Plates in bending

System | Moments | Shear forces



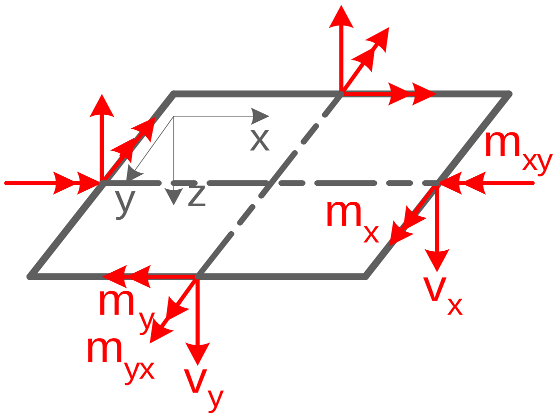
Sign convention for beams

Sign convention for plates in bending

Section forces and stresses in beams and plates in bending

Plates in bending

System



$$m_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \cdot z \cdot dz = \frac{h^2 \cdot \sigma_x^R}{6}$$

$$m_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y \cdot z \cdot dz = \frac{h^2 \cdot \sigma_y^R}{6}$$

Determination of the section forces of the plates in bending through integration of the stresses

$$\sigma_x^R = \frac{6}{h^2} \cdot m_x$$

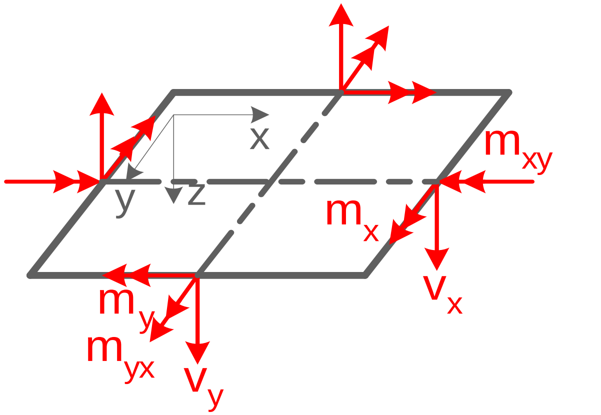
$$\sigma_y^R = \frac{6}{h^2} \cdot m_y$$

Indices: ´R´ - Edge stress in plates in bending

Section forces and stresses in beams and plates in bending

Plates in bending

System



$$m_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \cdot z \cdot dz = \frac{h^2 \cdot \tau_{xy}^R}{6}$$

$$m_{yx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yx} \cdot z \cdot dz = \frac{h^2 \cdot \tau_{yx}^R}{6}$$

Determination of the section forces of the plates in bending through integration of the stresses

$$\tau_{xy}^R = \frac{6}{h^2} \cdot m_{xy}$$

$$\tau_{yx}^R = \frac{6}{h^2} \cdot m_{yx}$$

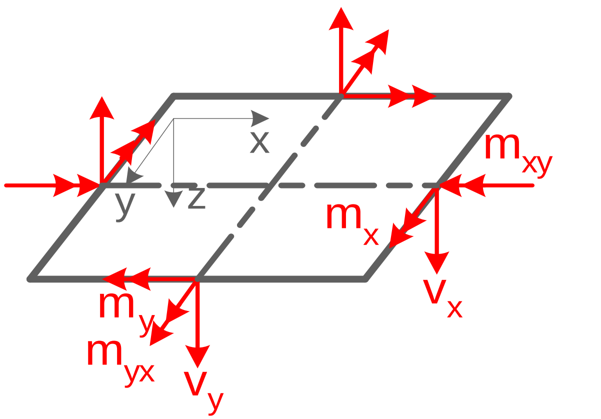
$$\tau_{yx} = \tau_{xy} \quad \rightarrow \quad m_{yx} = m_{xy}$$

Indices: ´R´ - Edge stress in plates in bending

Section forces and stresses in beams and plates in bending

Plates in bending

System



Determination of the section forces of the plates in bending through integration of the stresses

$$v_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \cdot dz = \frac{2h \cdot \tau^M_{xz}}{3}$$

$$v_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} \cdot dz = \frac{2h \cdot \tau^M_{yz}}{3}$$

$$\tau^M_{xz} = \frac{3}{2 \cdot h} \cdot v_x$$

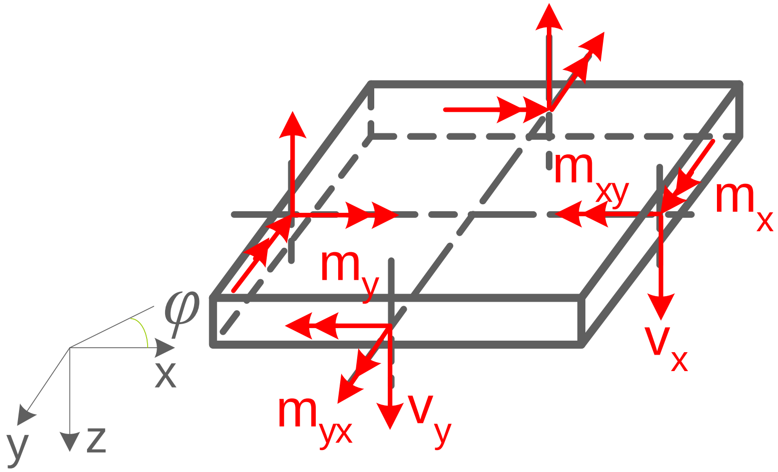
$$\tau^M_{yz} = \frac{3}{2 \cdot h} \cdot v_y$$

Indices: ´M´ - Edge stress in plates in bending

Principal bending moments in plates in bending

Principal stresses:

$$\sigma_{I,II} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



Principal bending moments in plates in bending

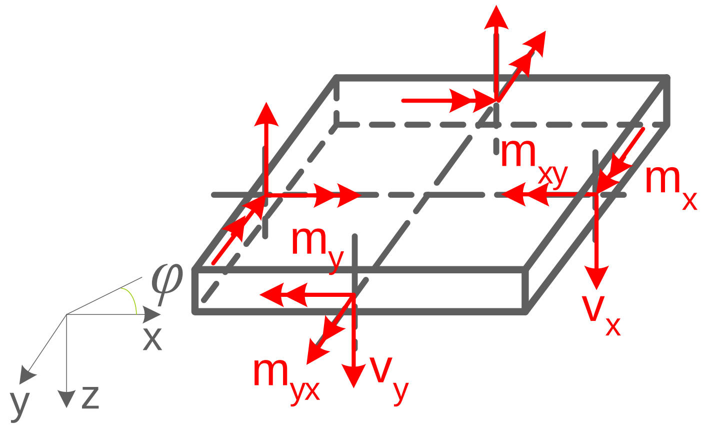
Difference between edge stresses and moments **m_x, m_y, m_{xy} : 6/h²**

$$m_x = \frac{h^2 \cdot \sigma_x^R}{6} \qquad m_y = \frac{h^2 \cdot \sigma_y^R}{6} \qquad m_{xy} = \frac{h^2 \cdot \tau_{xy}^R}{6}$$

Formula for principal stresses can also be applied for principal moments.

Principal bending moments:

$$m_{I,II} = \frac{m_x + m_y}{2} \pm \sqrt{\left(\frac{m_x - m_y}{2}\right)^2 + m_{xy}^2}$$



Principal directions:

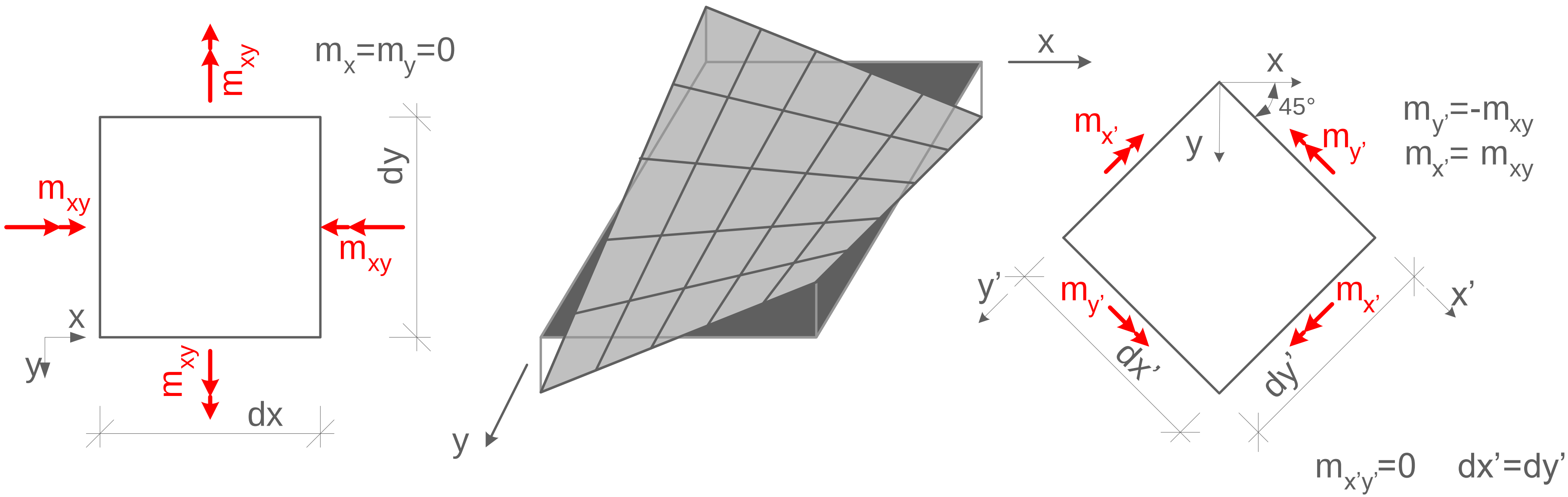
$$\tan(2\varphi) = \frac{2 \cdot m_{xy}}{m_x - m_y}$$

or

$$\tan(\varphi) = \frac{2 \cdot m_{xy}}{(m_x - m_y) + \sqrt{(m_x - m_y)^2 + 4 \cdot m_{xy}^2}}$$

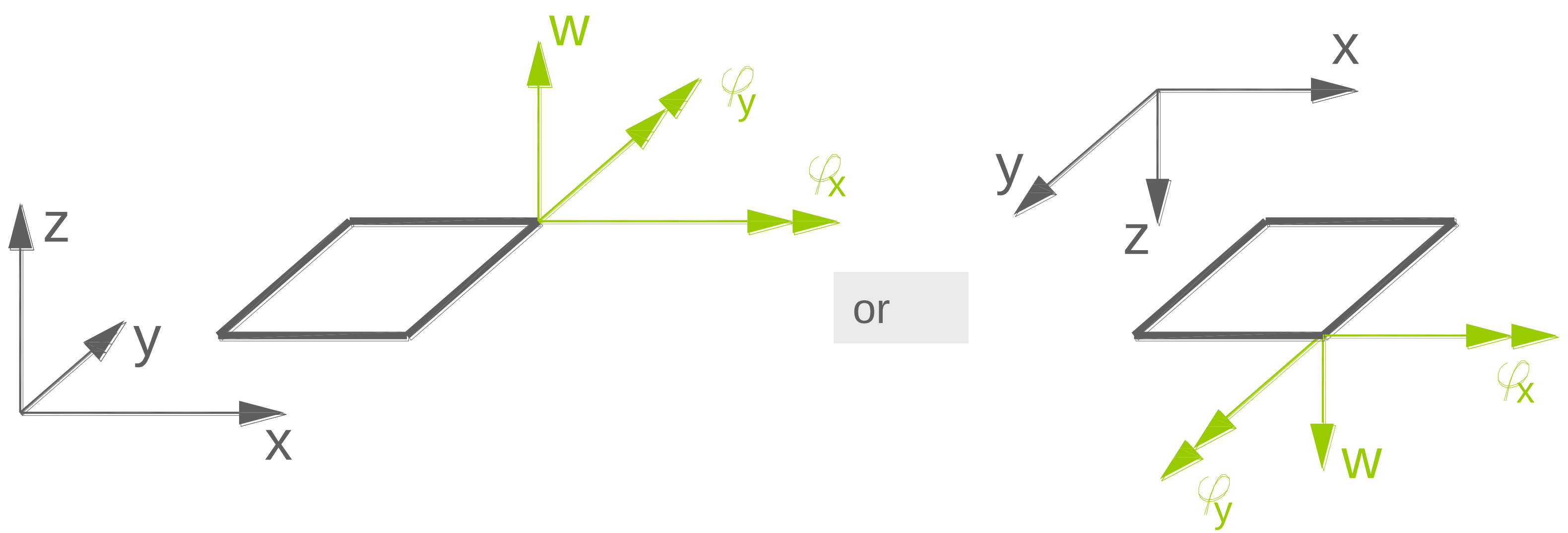
Principal bending moments in plates in bending

Deformations due to twisting moments



Twisting moments Deformed plate Equivalent system

Displacements w, φ_x, φ_y



Strains $\kappa_x, \kappa_y, \kappa_{xy} \left(\gamma_{xz}, \gamma_{yz}\right)$

Shear flexible plates: **Mindlin-Reissner plate theory**

Shear deformations are considered

$$\gamma_{xz}, \gamma_{yz}$$

Shear rigid plates: **Kirchhoff plate theory**

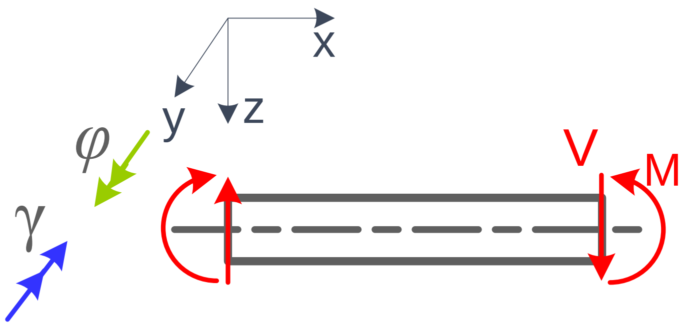
No shear deformation

$$\gamma_{xz} = \gamma_{yz} = 0 \quad \rightarrow \quad \varphi_x = \frac{dw}{dx}$$

$$\varphi_y = \frac{dw}{dy}$$

Strain forces in shear flexible plates

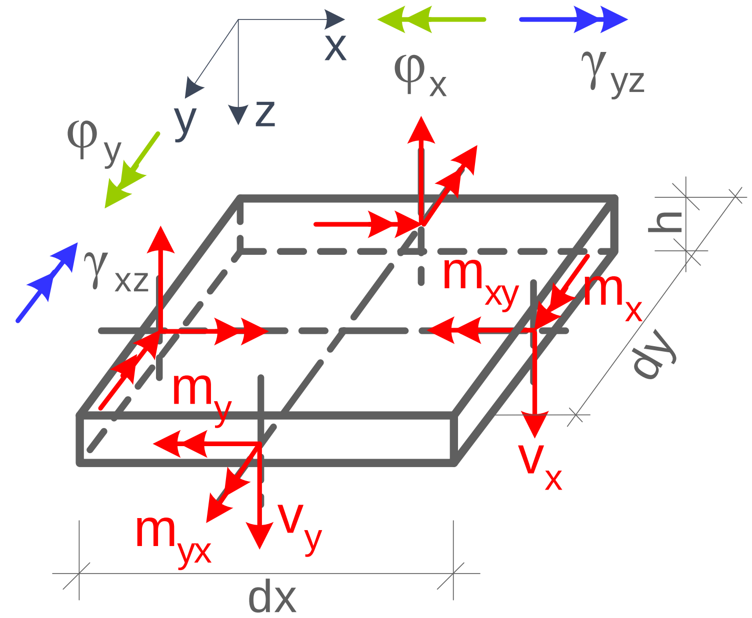
Beam



$$dw = -\varphi \cdot dx + \gamma \cdot dx$$

$$\rightarrow \quad \gamma = \varphi + \frac{dw}{dx}$$

Plate in bending

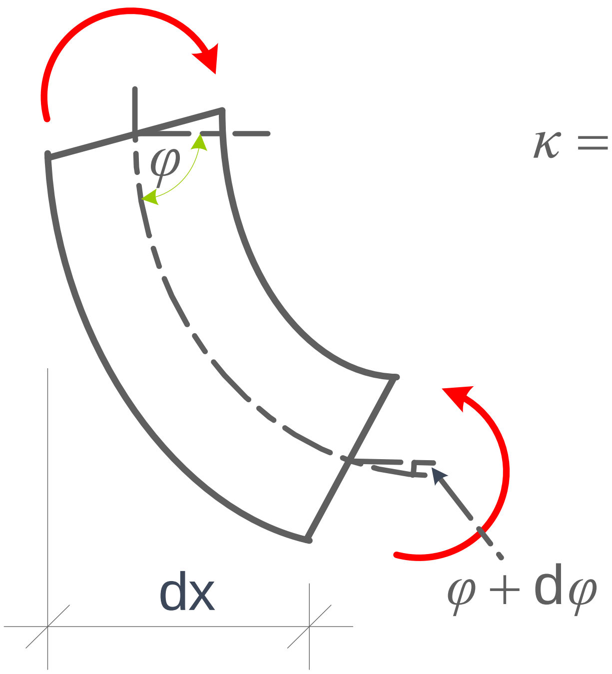


Strain forces in shear flexible plates

$$\kappa_x, \kappa_y, \kappa_{xy} \qquad \gamma_{xz}, \gamma_{yz}$$

Curvatures

Bending



$$\kappa = \frac{d\varphi}{dx}$$

$$\kappa_x = \frac{\partial \varphi_x}{\partial x}$$

Curvatures

$$\kappa_y = \frac{\partial \varphi_y}{\partial y}$$

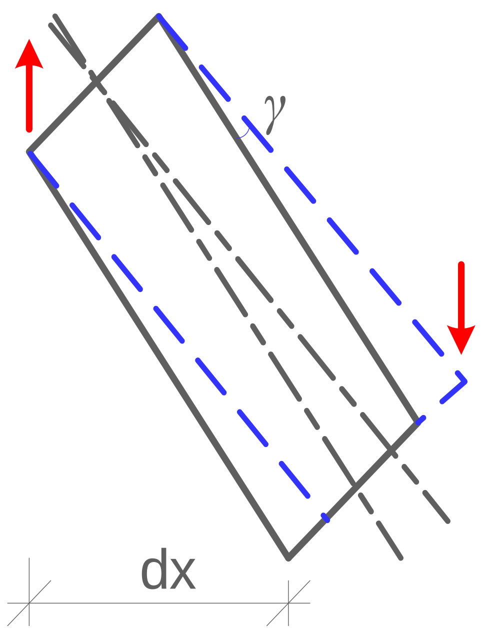
$$\kappa_{xy} = \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}$$

Torsion

Strain forces in shear flexible plates

$\kappa_x, \kappa_y, \kappa_{xy}$ γ_{xz}, γ_{yz}

Shear angle



Beam

$$\gamma = \varphi + \frac{dw}{dx}$$

Plate in bending

$$\gamma_{xz} = \varphi_x + \frac{dw}{dx}$$

$$\gamma_{yz} = \varphi_y + \frac{dw}{dy}$$

Section forces

Beam

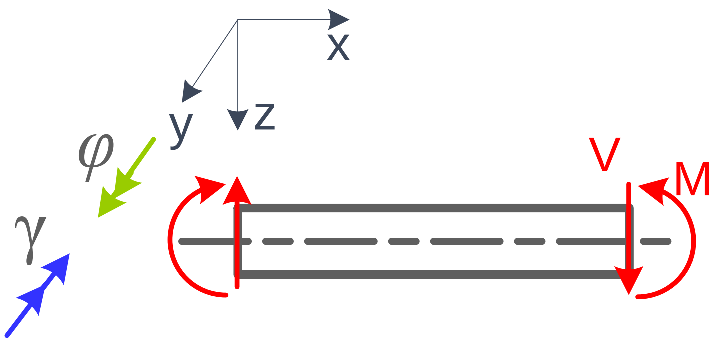
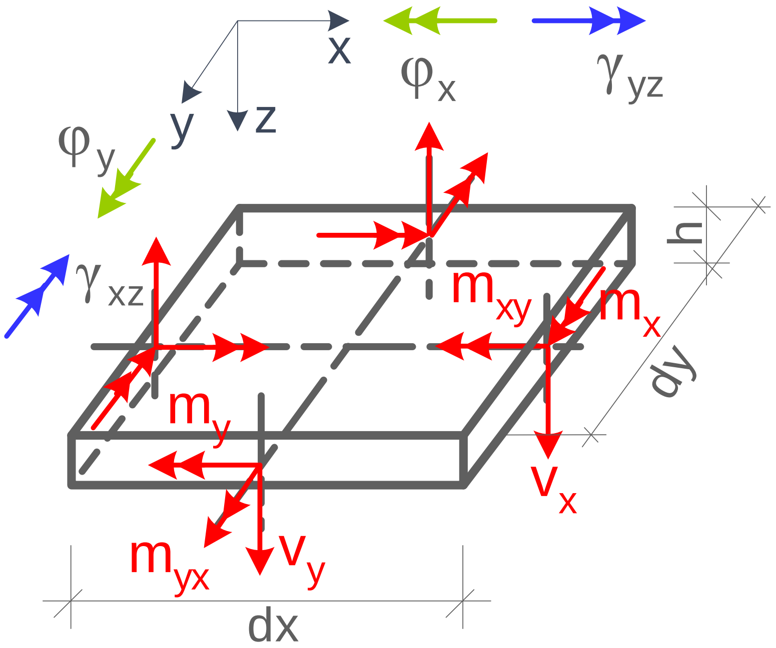


Plate in bending



Section forces : m_x, m_y, m_{xy}, v_x, v_y

Material law

„Moment – curvature“ relationship and „shear force – shear angle“ relationship

Beam

$M = EI \cdot \kappa$

$V = G \cdot A_S \cdot \gamma$

Plate in bending

$$\begin{bmatrix} m_x \\ m_y \\ m_{xy} \end{bmatrix} = \frac{Eh^3}{12(1-\mu^2)} \cdot \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \cdot \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$\underline{m} = \underline{D}_b \cdot \underline{\kappa}$$

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \frac{5 \cdot E \cdot h}{12(1+\mu)} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

$$\underline{v} = \underline{D}_s \cdot \underline{\gamma}$$

Equilibrium conditions

Equation of a shear rigid plate

(from the equilibrium conditions at an infinitesimal element)

$$\frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \cdot \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{K}$$

Plate stiffness $K = \frac{E \cdot h^3}{12(1-\mu^2)}$

$w(x,y)$ Bending $p(x,y)$ Distributed load

Section forces of a shear rigid plate

$$m_x = -K \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right)$$

$$m_y = -K \left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right)$$

$$m_{xy} = -(1-\mu) \cdot K \cdot \frac{\partial w^2}{\partial x \ \partial y}$$

$$v_x = \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y}$$

$$v_y = \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x}$$

Principle of virtual displacements

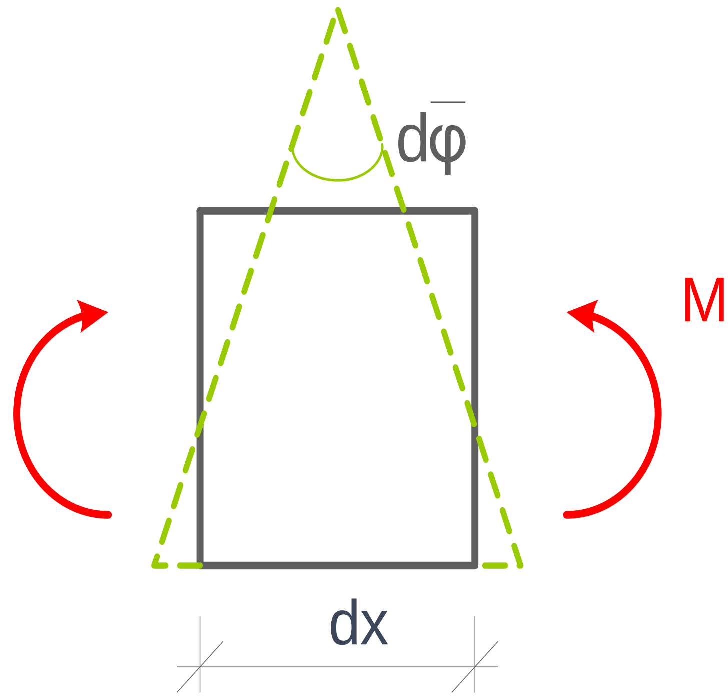
Internal virtual work

Shear flexible beam – contribution of the moment

$$d\overline{W}_i = M \cdot d\overline{\varphi}$$

$$\overline{\kappa} = \frac{d\overline{\varphi}}{dx} \rightarrow d\overline{\varphi} = \overline{\kappa} \cdot dx$$

$$d\overline{W}_i = M \cdot \overline{\kappa} \cdot dx$$



M = real moment

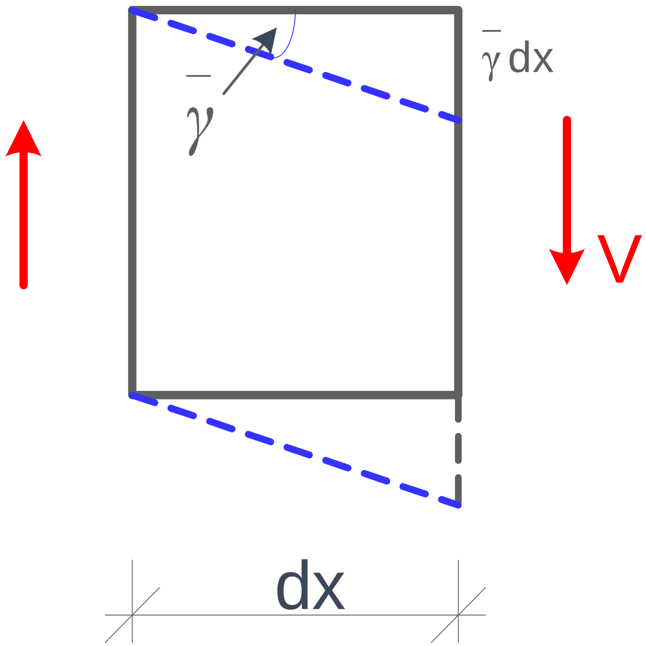
$\overline{\kappa}$ = virtual curvatures

Principle of virtual displacements

Internal virtual work

Shear flexible beam – contribution of the shear force

$$d\overline{W}_i = V \cdot \overline{\gamma} \cdot dx$$



V = real shear force

$\overline{\gamma}$ = virtual shear angle

Principle of virtual displacements

Internal virtual work

Shear flexible beam – total work of moment and shear force

$$\overline{W}_i = \int M \cdot \overline{\kappa} \cdot dx + \int V \cdot \overline{\gamma} \cdot dx$$

Shear flexible plate – total work of moment and shear force

$$\overline{W}_i = \int \begin{bmatrix} \overline{\kappa}_x & \overline{\kappa}_y & \overline{\kappa}_{xy} \end{bmatrix} \cdot \begin{bmatrix} m_x \\ m_y \\ m_{xy} \end{bmatrix} dx\ dy + \int \begin{bmatrix} \overline{\gamma}_{xz} & \overline{\gamma}_{yz} \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix} dx\ dy$$

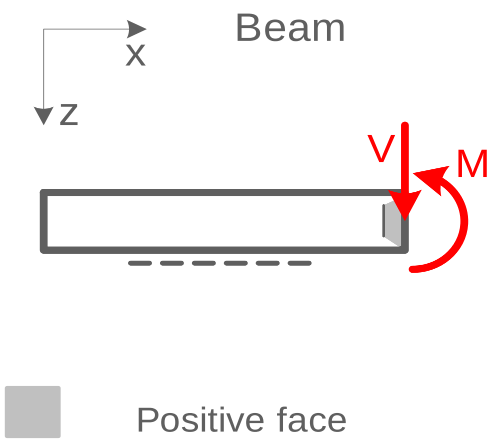
$$\overline{W}_i = \int \underline{\overline{\kappa}} \cdot \underline{m}\ dx\ dy + \int \underline{\overline{\gamma}} \cdot \underline{v}\ dx\ dy$$

End

1 Plates in plane stress

2 Plates in bending

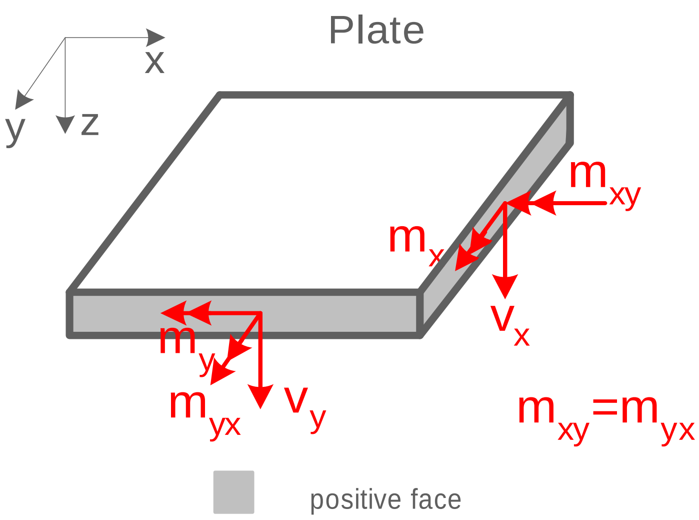
Sign convention of section forces in beams and plates in bending



Sign convention of section forces in Beams:

Section forces in beams and shear forces in plates are positive when they act in the positive coordinate direction at the positive face of an element.

Sign convention of section forces in beams and plates in bending



Sign convention of shear forces in plates:
Shear forces in plates are positive when they act in the positive coordinate direction at the positive face of an element.

Sign convention for moments in plate bending:
Bending moments m_x and m_y are positive if they lead to positive normal stresses σ_x and σ_y , respectively, at the bottom face of the plate. The bottom face of the plate is at the positive face in z-direction, i.e. on the positive z-face of the plate.
Twisting moments m_{xy} and m_{yx} are positive if they lead to shear stresses τ_{xy} and τ_{yx}, respectively, in the positive coordinate direction at the bottom face of the plate.